Stability estimates for the unique continuation property of the Stokes system. Application to an inverse problem *

Muriel Boulakia † Anne-Claire Egloffe ‡ Céline Grandmont ‡

December 3, 2012

Abstract

In the first part of this paper, we prove hölderian and logarithmic stability estimates associated to the unique continuation property for the Stokes system. The proof of these results is based on local Carleman inequalities. In the second part, these estimates on the fluid velocity and on the fluid pressure are applied to solve an inverse problem: we consider the Stokes system completed with mixed Neumann and Robin boundary conditions and we want to recover the Robin coefficient (and obtain stability estimate for it) from measurements available on a part of the boundary where Neumann conditions are prescribed. For this identification parameter problem, we obtain a logarithmic stability estimate under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed.

Keywords: Stability estimates, Local Carleman inequalities, Inverse boundary coefficient problem, Stokes system, Robin boundary conditions.

Mathematics Classification: 35B35, 35R30, 76D07.

1 Introduction

We are interested in stability estimates quantifying unique continuation properties for the Stokes system in a bounded connected domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, as well as their consequences for the stability of a Robin coefficient with respect to measurements available on one part of the boundary. Thus we will consider the Stokes system:

$$\begin{cases} -\Delta u + \nabla p &= 0, \quad \text{in } \Omega, \\ \operatorname{div} u &= 0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where u and p denote respectively the fluid velocity and the fluid pressure. For such a system and more generally for the unsteady Stokes equations with a non-smooth potential, C. Fabre and G. Lebeau proved in [FL96] a unique continuation result. In the particular case of the steady problem (1.1), their result is the following:

Theorem 1.1. *Let ω be an nonempty open set in Ω and $(u,p) \in H^1_{loc}(\Omega) \times L^2_{loc}(\Omega)$ be a weak solution of system* (1.1) *satisfying $u = 0$ in ω then $u = 0$ and p is constant in Ω.*

*This work was partially funded by the ANR-08-JCJC-013-01 (M3RS) project headed by C. Grandmont and the ANR-BLAN-0213-02 (CISIFS) project headed by L. Rosier.

†Université Pierre et Marie Curie-Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France & INRIA, Projet REO, Rocquencourt, BP 105, 78153 Le Chesnay cedex, France

‡INRIA, Projet REO, Rocquencourt, BP 105, 78153 Le Chesnay cedex, France & Université Pierre et Marie Curie-Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France.

We easily deduce from the previous theorem the following result (see [BEG]).

Corollary 1.2. *Let γ be a nonempty open set included in $\partial\Omega$ and $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ be a solution of system* (1.1) *satisfying $u = 0$ and $\frac{\partial u}{\partial n} - pn = 0$ on γ. Then $u = 0$ and $p = 0$ in Ω.*

One of the purposes of this paper is to obtain stability estimates in Ω quantifying the unique continuation result of Theorem 1.1 and Corollary 1.2 for any regular enough solution of the Stokes system and valid independently of the boundary conditions considered on $\partial\Omega$. In particular we will prove a local stability result which allows to estimate the velocity and the pressure on a compact set included in Ω. This inequality is of hölderian type:

Theorem 1.3. *Let ω be a nonempty open set and K be a compact set, both included in Ω. Then, there exists $c > 0$ and $0 < \beta < 1$ such that for all $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ solution of* (1.1)*, we have:*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq c \left(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}\right)^{\beta} \left(\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}\right)^{1-\beta}. \tag{1.2}$$

Then, we are going to prove two global logarithmic estimates. In the first one, we estimate (u,p) solution of (1.1) in the H^1-norm on the whole domain with respect to the L^2-norm of $(u_{|\Gamma}, p_{|\Gamma})$ and $\left(\frac{\partial u}{\partial n}_{|\Gamma}, \frac{\partial p}{\partial n}_{|\Gamma}\right)$, where Γ is a part of the boundary of Ω. In the second one, we obtain an estimate of (u,p) solution of (1.1) in the H^1-norm on the whole domain with respect to the H^1-norm of u and p in an open set $\omega \subset \Omega$. To be more specific, we prove the following theorem:

Theorem 1.4. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leq \frac{1}{2}$. Let Γ be a nonempty open subset of the boundary of Ω and ω be a nonempty open set included in Ω. Then, there exists $d_0 > 0$ such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, for all $\tilde{d} > d_0$, there exists $c > 0$, such that we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\ln\left(\tilde{d}\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\beta}}, \tag{1.3}$$

and

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\ln\left(\tilde{d}\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}}\right)\right)^{\beta}}, \tag{1.4}$$

for all couple $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (1.1).

We want to emphasize that these estimates are not optimal from the point of view of the unique continuation results stated previously. Indeed, one can notice that our stability estimates require more measurements than the Fabre–Lebeau unique continuation result. For instance, in Theorem 1.1, the unique continuation result only requires the velocity to be equal to zero whereas, in inequality (1.2), we need information on u and p on ω. One can refer to [LUW10] where an optimal three balls inequality which only involves the velocity u is obtained in L^2-norm. Moreover, note that the constraint $\frac{\partial u}{\partial n} - pn$ which appears in Corollary 1.2 is divided into two terms in inequality (1.3): $\frac{\partial u}{\partial n}$ in one hand and pn in the other hand and that there is also an additional term, the normal derivative of p. Nevertheless, even if these estimates are not optimal, they are satisfied without prescribing boundary conditions satisfied by the solution and have the advantage of providing an upper bound both on u and p. This point will be crucial to solve the inverse problem of identifying a Robin coefficient defined on some part of the boundary from measurements available on another part of the boundary.

To prove these results, we will follow the same steps as in [Phu03], where K. D. Phung has obtained a quantitative uniqueness result for the Laplace equation. The proof is based on local Carleman inequalities (inside the domain and near the boundary) coming from pseudo-differential calculus. Note that it requires the domain to be of class $\mathcal{C}^\infty$. In [BD10], L. Bourgeois and J. Dardé extend the estimate proved by in [Phu03] to Lipschitz domains. For such non smooth domains, difficulties occur when one wants to estimate the function in a neighborhood of $\partial\Omega$: the authors use interior Carleman estimate and a technique based on a sequence of balls which approaches the boundary, which is inspired by [ABRV00]. Let us emphasize the fact that the inequality obtained by this way is valid for a regular solution u (u belongs to $\mathcal{C}^{1,\alpha}(\Omega)$ and is such that $\Delta u \in L^2(\Omega)$).

A second objective of this paper is to apply the previous stability estimates to some parameter identification problem: we assume that mixed Neumann and Robin conditions are prescribed and our aim is to derive stability estimates for the inverse problem of determining the Robin coefficient from measurements available on a part of the boundary where Neumann boundary conditions are prescribed. More precisely, we introduce the following boundary problem:

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega \\ div\ u & = & 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}. \end{array}\right. \tag{1.5}$$

Our objective is to determine the coefficient q from the values of u and p on Γ_{out}. Such kind of systems naturally appears in the modeling of biological problems like, for example, blood flow in the cardiovascular system (see [QV03] and [VCFJT06]) or airflow in the lungs (see [BGM10]). For an introduction on the modeling of the airflow in the lungs and on different boundary conditions which may be prescribed, we refer to [Egl12]. The part of the boundary Γ_0 represents a physical boundary on which measurements are available and Γ_{out} represents an artificial boundary on which Robin boundary conditions or mixed boundary conditions involving the fluid stress tensor and its flux at the outlet are prescribed. For this problem, we will prove a logarithmic estimate under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed. This later assumption can be discarded in very specific cases (see [BEG]) and is generally veryfied numerically in the considered applications.

Stability estimates for the Robin coeffficient have been widely studied for the Laplace equation [ADPR03], [BCC08], [CFJL04], [CJ99], [CCL08] and [Sin07]. This kind of problems arises in general in corrosion detection which consists in determining a Robin coefficient on the inaccessible portion of the boundary thanks to electrostatic measurements performed on the accessible boundary. Most of these papers prove a logarithmic stability estimate ([ADPR03], [BCC08], [CFJL04] and [CCL08]). We mention that, in [CJ99], S. Chaabane and M. Jaoua obtained both local and monotone global Lipschitz stability for regular Robin coefficient and under the assumption that the flux g is non negative. Under the *a priori* assumption that the Robin coefficient is piecewise constant, E. Sincich has obtained in [Sin07] a Lipschitz stability estimate. To prove stability estimates, different approaches are developed in these papers. A first approach consists in using the complex analytic function theory (see [ADPR03], [CFJL04]). A characteristic of this method is that it is only valid in dimension 2. Another classical approach is based on Carleman estimates (see [BCC08] and [CCL08]). In [BCC08], the authors use the result proved by K.D. Phung in [Phu03] to obtain a logarithmic stability estimate which is valid in any dimension for an open set Ω of class $\mathcal{C}^\infty$. Moreover, in [BCC08], the authors use semigroup theory to obtain a stability estimate in long time for the heat equation from the stability estimate for the Laplace equation.

The inverse problem of recovering Robin coefficients for the Stokes system has already been studied in [BEG] where we have obtained a logarithmic stability estimate valid in dimension 2 for the steady problem as well as the unsteady one, under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed. An improvement of the present paper is that the stability estimate is valid in any space dimension. Moreover, if we compare the result stated in Theorem 4.3 in the particular case $d = 2$ with the previous result in [BEG], we realize that we need less regularity on the solution (u, p) in Theorem 4.3. To be more precise, in [BEG], the solution (u, p) has to belong to $H^4(\Omega) \times H^3(\Omega)$ whereas, here, the regularity in $H^3(\Omega) \times H^2(\Omega)$ is sufficient. Another improvement lies in the fact that the power of the logarithm involved in the stability estimate (4.2) of Theorem 4.3 is better than the one obtained in [BEG]: the power is equal to $\frac{3}{4}\beta$ here, whereas it was equal to $\frac{\beta}{2}$ in [BEG], for all $\beta \in (0, 1)$.

Let us describe the content of the paper. The second section is dedicated to the statement of Carleman inequalities. Adapting the method of [Phu03], we will use these Carleman inequalities to prove Theorems 1.3 and 1.4 in the third section. The proof of Theorem 1.4 is divided into three intermediate results which illustrate how the information spreads from a part of the boundary to another, whereas the proof of Theorem 1.3 is a direct consequence of one of the intermediate results. The fact that, in the right-hand side of inequality (1.2), we only need the L^2-norm of p is due to Caccioppoli inequality. As in [Phu03], we use two kinds of local Carleman inequalities, one near the boundary and one in the interior of the open set Ω. In each case, it consists in applying simultaneously the Carleman estimate to u and p, by using the fact that $\Delta u = \nabla p$ and $\Delta p = \operatorname{div}(\Delta u) = 0$, in order to free ourselves from terms in the right-hand side of the inequalities. It is interesting to note that if we directly apply estimate coming from [Phu03] to (u, p) solution of Stokes equations, and if we perform the same reasoning as explained above, we obtain ∇p in L^2 norm over all Ω in the right-hand side of the inequality which we can not discard. Consequently, we can not prove Theorems 1.3 and 1.4 without going deeply in the heart of the proof. Finally, in section 4, in the same spirit as in [BCC08], we use inequality of Theorem 1.4 to obtain a logarithmic stability estimate of a Robin coefficient on one part of the boundary for (u, p) solution of the Stokes problem with respect to the trace of u and p available on another part of the boundary.

If not specified otherwise, c is a generic constant, whose value may change and which only depends on the geometry of the open set Ω. Moreover, we denote indifferently by $|\ |$ a norm on $\mathbb{R}^n$, for any $n \geq 1$.

For $x = (x_1, ..., x_d) \in \mathbb{R}^d$, we denote by $x' \in \mathbb{R}^{d-1}$ the $d-1$ first coordinates of x. We will also use the following notation: $\mathbb{R}^d_+ = \{x = (x', x_d) \in \mathbb{R}^d / x_d \geq 0\}$.

2 Local Carleman inequalities

In this section, we recall local Carleman inequalities, firstly inside Ω, then on the boundary of Ω. These inequalities are based on Gårding inequality, which is itself a consequence of pseudo-differential calculus.

Definition 2.1. *Let $h > 0$, P be an operator and $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$. Let us define the conjugate operator $P_\phi = -h^2 e^{\phi/h} \circ P \circ e^{-\phi/h}$ and p_ϕ its principal symbol. We recall that the Poisson bracket between Re p_ϕ and Im p_ϕ is defined by:*

$$\{Re\ p_\phi, Im\ p_\phi\} = \nabla_\xi Re\ p_\phi \nabla_x Im\ p_\phi - \nabla_x Re\ p_\phi \nabla_\xi Im\ p_\phi.$$

We say that ϕ satisfies the Hörmander hypoellipticity property on K if:

$$\exists\, c_1 > 0,\ \forall\, (x, \xi) \in K \times \mathbb{R}^d, p_\phi(x, \xi) = 0 \Rightarrow \{Re\ p_\phi, Im\ p_\phi\}(x, \xi) \geq c_1. \tag{2.1}$$

Proposition 2.2. *Let U be an open set in $\mathbb{R}^d$ and K be a compact set included in U. Let us consider $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $P = \Delta$. We assume that the function ϕ satisfies*

$$|\nabla \phi| > 0 \text{ in } U,$$

and the Hörmander hypoellipticity property on U given by (2.1).

Then, there exists $c > 0$ and $h_1 > 0$ such that for all $h \in (0, h_1)$ and for all function $y \in \mathcal{C}_0^\infty(K)$, we have

$$\int_K |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\nabla y(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_K |\Delta y(x)|^2 e^{2\phi(x)/h} dx.$$

Proof of Proposition 2.2. We refer to [Hör85] for a proof of this inequality. □

Remark 2.3. *We can extend the previous inequality to functions which belong to $\{y \in H_0^1(K) / \Delta y \in L^2(K)\}$ by a density argument.*

Locally near the boundary, we can go back to the half-space by a change of coordinates. We have the following Carleman inequality:

Proposition 2.4. *Let $K = \{x \in \mathbb{R}^d_+ / |x| \leq R_0\}$ and $\Sigma = \{x \in \partial K / x_d = 0\}$. Let us denote by $\mathcal{C}^\infty_{0,\partial K\setminus\Sigma}(\overline{K})$ the restriction to K of $\mathcal{C}^\infty_0(\overline{B(0,R_0)})$ functions.*

Let P be a second-order differential operator whose coefficients are $\mathcal{C}^\infty$ in a neighborhood of K, defined by $P(x,\partial_x) = -\partial^2_{x_d} + R(x, \frac{1}{i}\partial_{x'})$ and ϕ be a $\mathcal{C}^\infty$ function defined in a neighborhood of K. Let us denote by $r(x,\xi')$ the principal symbol of R and assume that $r(x,\xi') \in \mathbb{R}$ and that there exists a constant $c > 0$ such that $(x,\xi') \in K \times \mathbb{R}^{d-1}$, we have $r(x,\xi') \geq c|\xi'|^2$.

We assume that the function ϕ satisfies (2.1) and

$$\partial_{x_d}\phi(x) \neq 0, \, \forall \, x \in K.$$

Then, there exists $c > 0$ and $h_1 > 0$ such that for all $h \in (0,h_1)$ and for all function $y \in \mathcal{C}^\infty_{0,\partial K\setminus\Sigma}(\overline{K})$, we have:

$$\begin{aligned}\int_{\mathbb{R}^d_+} |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\mathbb{R}^d_+} |\nabla y(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_{\mathbb{R}^d_+} |P(x,\partial_x)y(x)|^2 e^{2\phi(x)/h} dx \\ + c\int_{\mathbb{R}^{d-1}} (|y(x',0)|^2 + |h\partial_{x'}y(x',0)|^2 + |h\partial_{x_d}y(x',0)|^2) e^{2\phi(x',0)/h} dx'.\end{aligned}$$

Proof of Proposition 2.4. We refer to [LR95] for a proof of this inequality. □

Remark 2.5. *Let us denote by $H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K)$ the restriction to the set K of functions in $H^{\frac{3}{2}+\nu}_0(B(0,R_0))$. We can extend the previous inequality to functions which belong to $H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K)$ by a density argument.*

The key point to apply the previous Propositions consists in the construction of a function ϕ which satisfies the Hörmander hypoellipticity property (2.1). The two following lemmas are proved in [Phu03]. The first one gives an example of function which satisfies the Hörmander hypoellipticity property inside an open set.

Lemma 2.6. *Let $0 < \delta < M$, $\lambda > 0$ and $q \in \mathbb{R}^d$. The function $\phi(x) = e^{-\lambda|x-q|^2}$ satisfies (2.1) on the set $K = \{(x,\xi) \in \mathbb{R}^d / \delta < |x-q| < M\}$ as soon as λ is large enough.*

The following lemma gives us functions which satisfies the Hörmander hypoellipticity property near the boundary.

Lemma 2.7. *Let $\lambda > 0$ and $R_0 > 0$. We denote by $K = \{x \in \mathbb{R}^d_+ / |x| \leq R_0\}$.*

Then, $\phi(x) = e^{\lambda x_d}$ satisfies (2.1) on K as soon as λ is large enough. Moreover, the functions

- $\phi(x) = e^{-\lambda x_d}$,
- $\phi(x) = e^{-\lambda(x_d + |x|^2)}$,

also satisfy (2.1) on K as soon as λ is large enough and R_0 is small enough.

We end this section by a lemma which will be useful in the following.

Lemma 2.8. *Let P be a second-order differential operator defined in an open set M and $\chi \in \mathcal{C}^\infty_0(M)$ such that $\chi = 1$ in a subdomain Π of M. Then, $P(\chi y) = \chi P y + [P,\chi]y$ with $[P,\chi]$ a first-order operator with support in $M\setminus\Pi$. Moreover, there exists $c > 0$ such that for all $y \in H^1(M)$,*

$$\|[P,\chi]y\|_{L^2(M)} \leq c\|y\|_{H^1(M\setminus\Pi)}.$$

3 Stability estimates

In this section, we give a proof of Theorems 1.3 and 1.4.

3.1 Main results

In this first subsection, we state some intermediate results. We first introduce two theorems, Theorem 3.1 and Theorem 3.2, and we prove that Theorem 1.3 and Theorem 1.4 are respectively equivalent to Theorem 3.1 and Theorem 3.2. Next, we state Propositions 3.6, 3.8 and 3.9 which will allow to prove Theorems 3.1 and 3.2.

Theorem 3.1. *Let ω be a nonempty open set and K be a compact set, both included in Ω. Then, there exists $c > 0$ and $s > 0$ such that for all $(u, p) \in H^1(\Omega) \times L^2(\Omega)$ solution of (1.1), and for all $\epsilon > 0$ we have:*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}\right). \tag{3.1}$$

Theorem 3.2. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leq \frac{1}{2}$, Γ be a nonempty open subset of the boundary of Ω and ω be a nonempty open set included in Ω. Then, for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$, we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}}\left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \tag{3.2}$$

and

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \tag{3.3}$$

for all couple $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (1.1).

Proposition 3.3. *Theorem 3.1 and Theorem 1.3 are equivalent.*

Proof of Proposition 3.3. The fact that Theorem 3.1 implies Theorem 1.3 is a direct consequence of Lemma 3.4 below with

$A = c\left(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}\right)$, $B = \|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}$, $C_1 = 1$, $C_2 = s$ and $D = \|u\|_{H^1(K)} + \|p\|_{L^2(K)}$.

Moreover, the fact that Theorem 1.3 implies Theorem 3.1 is a consequence of Young inequality by writing:

$$c\left(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}\right)^\beta \left(\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}\right)^{1-\beta} = \left(\frac{c}{\epsilon}(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)})\right)^\beta \left(\epsilon^{\frac{\beta}{1-\beta}}(\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)})\right)^{1-\beta}.$$

□

Lemma 3.4. *Let $A > 0$, $B > 0$, $C_1 > 0$, $C_2 > 0$ and $D > 0$. We assume that there exists $c_0 > 0$ and $\gamma_1 > 0$ such that $D \leq c_0 B$ and for all $\gamma \geq \gamma_1$,*

$$D \leq Ae^{C_1\gamma} + Be^{-C_2\gamma}. \tag{3.4}$$

Then, there exists $C > 0$ such that:

$$D \leq CA^{\frac{C_2}{C_1+C_2}} B^{\frac{C_1}{C_1+C_2}}.$$

Proof of Lemma 3.4. Let $\gamma_0 = \dfrac{1}{C_1 + C_2} \ln\left(\dfrac{B}{A}\right)$. Two cases arise:

- if $\gamma_0 \geq \gamma_1$, we directly obtain the desired result by applying inequality (3.4) with $\gamma = \gamma_0$,
- if $\gamma_0 < \gamma_1$ then $B < e^{(C_1+C_2)\gamma_1} A$, which implies the desired inequality since by assumption $D \leq cB$.

$\square$

Proposition 3.5. *Theorem 3.2 and Theorem 1.4 are equivalent.*

Proof of Proposition 3.5. Let us prove the equivalence between inequality (1.3) of Theorem 1.4 and inequality (3.2) of Theorem 3.2, the equivalence between inequalities (1.4) and (3.3) can be proved in the same way.

We denote by

$$A = \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \quad \text{and} \quad B = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\tfrac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\tfrac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}.$$

Assume that inequality (3.2) is true. By continuity of the trace mapping, we know that there exists a constant $d_0 > 0$ such that $B \leq A d_0$. Thus, for all $\tilde{d} > d_0$, we have $\dfrac{\tilde{d}A}{B} > 1$. By choosing $\epsilon = \dfrac{c}{1-\beta}\left(\ln\left(\dfrac{\tilde{d}A}{B}\right)\right)^{-1}$, then replacing in (3.2), we obtain the existence of $C > 0$ such that:

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq CA\left(\left(\frac{B}{\tilde{d}A}\right)^{\beta} + \frac{1}{\left(\ln\left(\frac{\tilde{d}A}{B}\right)\right)^{\beta}}\right).$$

Then we use the fact that for all $x > 1$, $\dfrac{1}{x} \leq \dfrac{1}{\ln(x)}$ to conclude.

Reciprocally, assume that inequality (1.3) holds true. Thus, for all $\beta \in \left(0, \dfrac{1}{2} + \nu\right)$, for all $\tilde{d} > d_0$, there exists $c > 0$, for all couple $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$, solution of (1.1)

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c\frac{A}{\left(\ln\left(\frac{\tilde{d}A}{B}\right)\right)^{\beta}}. \tag{3.5}$$

This implies that:

$$\frac{\tilde{d}A}{B} \leq \exp\left(\left(\frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}}\right)^{\frac{1}{\beta}}\right). \tag{3.6}$$

Let $\epsilon > 0$. We will consider the two following cases

$$\left(\frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}}\right)^{\frac{1}{\beta}} \leq \frac{1}{\epsilon} \tag{3.7}$$

and

$$\left(\frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}}\right)^{\frac{1}{\beta}} > \frac{1}{\epsilon} \tag{3.8}$$

separately to obtain inequality (3.2). If inequality (3.7) is satisfied, then (3.6) implies that $\dfrac{\tilde{d}A}{B} \leq e^{\frac{1}{\epsilon}}$ and we conclude by using the fact that $H^{\frac{3}{2}+\nu}(\Omega) \hookrightarrow H^1(\Omega)$. If inequality (3.8) is satisfied, we obtain directly:

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} < cA\epsilon^{\beta}, \tag{3.9}$$

which allows us to conclude. $\square$

Let us now state three propositions. Theorem 3.2 will be proved using these three intermediate results whereas Theorem 3.1 will be a consequence of inequality (3.11) of Proposition 3.6. The first proposition allows to transmit information from an open set to any relatively compact open set in Ω.

Proposition 3.6. *Let ω be a nonempty open set included in Ω and let $\hat{\omega}$ be a relatively compact open set in Ω. Then:*

$$\left\{\begin{array}{l}\exists\, c, s>0, \forall\, \epsilon>0, \forall\,(u,p) \in H^1(\Omega)\times H^1(\Omega) \text{ solution of } (1.1),\\ \|u\|_{H^1(\hat{\omega})}+\|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)}+\|p\|_{H^1(\omega)}\right)+\epsilon^s\left(\|u\|_{H^1(\Omega)}+\|p\|_{H^1(\Omega)}\right),\end{array}\right. \tag{3.10}$$

and

$$\left\{\begin{array}{l}\exists\, c, s>0, \forall\, \epsilon>0, \forall\,(u,p) \in H^1(\Omega)\times L^2(\Omega) \text{ solution of } (1.1),\\ \|u\|_{H^1(\hat{\omega})}+\|p\|_{L^2(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)}+\|p\|_{L^2(\omega)}\right)+\epsilon^s\left(\|u\|_{H^1(\Omega)}+\|p\|_{L^2(\Omega)}\right).\end{array}\right. \tag{3.11}$$

Remark 3.7. *Note that the difference between inequalities* (3.10) *and* (3.11) *lies in the fact that in* (3.11) *we only have the L^2-norm of p instead of the H^1-norm in each hand-side.*

The second proposition allows to transmit information from a relatively compact open set in Ω to a neighborhood of the boundary.

Proposition 3.8. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0<\nu\leq\frac{1}{2}$. Let $x_0\in\partial\Omega$ and let ω be an open set in Ω. There exists a neighborhood $\hat{\omega}$ of x_0 such that:*

$$\forall\,\beta\in\left(0,\frac{1}{2}+\nu\right), \exists\, c>0, \forall\,\epsilon>0, \forall\,(u,p)\in H^{\frac{3}{2}+\nu}(\Omega)\times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of } (1.1),$$

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)}+\|p\|_{H^1(\hat{\omega}\cap\Omega)} \leq e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)}+\|p\|_{H^1(\omega)})+\epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)}+\|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \tag{3.12}$$

Finally, the third proposition allows to transmit information from a part of the boundary of Ω to a relatively compact open set in Ω.

Proposition 3.9. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0<\nu\leq\frac{1}{2}$. Let Γ be a nonempty open subset of the boundary of Ω. Let $\hat{\omega}$ be a relatively compact open set in Ω. Then, we have the following assertion:*

$$\exists\, c, s>0, \forall\,\epsilon>0, \forall\,(u,p)\in H^{\frac{3}{2}+\nu}(\Omega)\times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of } (1.1),$$

$$\|u\|_{H^1(\hat{\omega})}+\|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)}+\|p\|_{H^1(\Gamma)}+\left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)}+\left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right)+\epsilon^s(\|u\|_{H^1(\Omega)}+\|p\|_{H^1(\Omega)}).$$

Remark 3.10. *The logarithmic nature of inequalities* (3.2) *and* (3.3) *comes from Proposition 3.8 where an exponential appears in front of the first term of the right-hand side whereas the estimates in Propositions 3.6 and 3.9 lead to hölderian estimates, as a consequence of Lemma 3.4.*

Remark 3.11. *In Proposition 3.9, the regularity of u and p in $H^{\frac{3}{2}+\nu}$ is necessary to give a sense to the normal derivatives.*

The next subsection is dedicated to the proof of Propositon 3.6. In the third subsection, we prove Propositions 3.8 and 3.9. Finally, in the last subsection, we conclude with the proof of the main theorems.

3.2 Estimates on relatively compact open sets: proof of Proposition 3.6

Let us begin by a lemma which will be useful to prove Proposition 3.6.

Notation 3.12. *Let $q\in\mathbb{R}^d$ and $0<r<r'$. We denote by $A_q(r,r')$ the annulus delimited by the area between two concentric circles of centre q and of respective radius r and r':*

$$A_q(r,r')=\{x\in\mathbb{R}^d/r<|x-q|<r'\}.$$

Lemma 3.13. *Let $q \in \mathbb{R}^d$ and $0 < r_1 < r_2 < r_3 < r_4 < r_5$. Then, there exists $c > 0$, $h_1 > 0$, $c_1 > 0$ and $c_2 > 0$ such that for all $0 < h < h_1$ and for all function $(u,p) \in H^1(B(q,r_5)) \times H^1(B(q,r_5))$ solution of*

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p &=& 0, & \text{in } B(q,r_5), \\ div \ \ u &=& 0, & \text{in } B(q,r_5), \end{array} \right. \tag{3.13}$$

the following inequality is satisfied :

$$\begin{aligned} &\|u\|_{H^1(A_q(r_2,r_3))} + \|p\|_{H^1(A_q(r_2,r_3))} \\ &\quad \leq c \left(e^{c_1/h} (\|u\|_{H^1(B(q,r_2))} + \|p\|_{H^1(B(q,r_2))}) + e^{-c_2/h} (\|u\|_{H^1(B(q,r_5))} + \|p\|_{H^1(B(q,r_5))}) \right), \end{aligned} \tag{3.14}$$

with $c_1 = g(r_1) - g(r_3) > 0$ and $c_2 = g(r_3) - g(r_4) > 0$, where $g(x) = e^{-\lambda x^2}$ and λ large enough.

Proof of Lemma 3.13. Let r_0 and r_6 be such that $0 < r_0 < r_1$ and $r_5 < r_6$. We are going to apply Proposition 2.2 with

$$U_0 = A_q(r_0, r_6), \quad K_0 = \overline{A_q(r_1, r_5)},$$

and $\phi(x) = e^{-\lambda|x-q|^2}$ for λ large enough, so that assumptions of Proposition 2.2 are verified according to Lemma 2.6.

Let $\chi \in \mathcal{C}_c^\infty(B(q,r_6))$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $\Pi = A_q(r_2,r_4)$ and $\chi = 0$ in the exterior of K_0. Note that:

$$\Pi \subset K_0 \subset U_0.$$

Thanks to Remark 2.3, we can apply Proposition 2.2 successively to χu and χp where (u,p) is solution of (3.13): there exists $c > 0$ and $h_1 > 0$ such that for all $h \in (0,h_1)$ and for all function $(u,p) \in H^1(B(q,r_5)) \times H^1(B(q,r_5))$ solution of (3.13), we have, thanks to Lemma 2.8:

$$\begin{aligned} &\int_\Pi |u(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_\Pi |\nabla u(x)|^2 e^{2\phi(x)/h} dx \\ &\qquad \leq ch^3 \int_{K_0} |\chi \nabla p(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi] u(x)|^2 e^{2\phi(x)/h} dx, \end{aligned} \tag{3.15}$$

and since $\Delta p = \operatorname{div}(\Delta u) = 0$:

$$\int_\Pi |p(x)|^2 e^{2\phi(x)/h} dx \ + \ h^2 \int_{K_0} |\nabla(\chi p)(x)|^2 e^{2\phi(x)/h} dx \quad \leq \quad ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi] p(x)|^2 e^{2\phi(x)/h} dx.$$

Note that:

$$|\nabla(\chi p)(x)|^2 = |\chi(x) \nabla p(x) + p(x) \nabla \chi(x)|^2 = |\chi(x) \nabla p(x)|^2 + |p(x) \nabla \chi(x)|^2 + 2\chi(x) \nabla p(x) \cdot p(x) \nabla \chi(x).$$

Using Cauchy-Schwarz inequality and Young inequality, we have:

$$2\chi(x) \nabla p(x) \cdot p(x) \nabla \chi(x) \geq -2 |\chi(x) \nabla p(x)| |p(x) \nabla \chi(x)| \geq -\frac{|\chi(x) \nabla p(x)|^2}{2} - 8 |p(x) \nabla \chi(x)|^2.$$

Thus, it follows:

$$\begin{aligned} &\int_\Pi |p(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{K_0} |\chi \nabla p(x)|^2 e^{2\phi(x)/h} dx \\ &\qquad \leq ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi] p(x)|^2 e^{2\phi(x)/h} dx + ch^2 \int_{K_0 \setminus \Pi} |p(x)|^2 e^{2\phi(x)/h} dx. \end{aligned} \tag{3.16}$$

We add up inequalities (3.15) and (3.16): there exists $h_1 > 0$ such that for all $h \in (0, h_1)$,

$$\begin{aligned} &e^{g(r_3)/h} \int_{A_q(r_2,r_3)} (|u(x)|^2 + |p(x)|^2 + h^2 (|\nabla u(x)|^2 + |\nabla p(x)|^2) dx \\ &\qquad \leq ch^2 e^{g(r_1)/h} \int_{A_q(r_1,r_2)} |[\Delta, \chi] u(x)|^2 + |[\Delta, \chi] p(x)|^2 + |p(x)|^2 dx \\ &\qquad\qquad + ch^2 e^{g(r_4)/h} \int_{A_q(r_4,r_5)} |[\Delta, \chi] u(x)|^2 + |[\Delta, \chi] p(x)|^2 + |p(x)|^2 dx. \end{aligned}$$

We divide the previous inequality by h^2. Using again Lemma 2.8, we obtain that there exists $c > 0$, $h_1 > 0$, $c_1 = g(r_1) - g(r_3) > 0$ and $c_2 = g(r_3) - g(r_4) > 0$ such that for all $h \in (0, h_1)$ and for all function $(u, p) \in H^1(B(q, r_5)) \times H^1(B(q, r_5))$ solution of (3.13), we have

$$\begin{aligned}
&\|u\|_{H^1(A_q(r_2,r_3))} + \|p\|_{H^1(A_q(r_2,r_3))} \\
&\quad \le c\left(e^{c_1/h}(\|u\|_{H^1(B(q,r_2))} + \|p\|_{H^1(B(q,r_2))}) + e^{-c_2/h}(\|u\|_{H^1(B(q,r_5))} + \|p\|_{H^1(B(q,r_5))})\right).
\end{aligned}$$

□

Let us introduce the notion of δ-sequence of balls between two points.

Definition 3.14. *Let $\delta > 0$ and (x_0, x) be two points in Ω. We say that $(B(q_j, \delta))_{j=0,\dots,N}$ is a δ-sequence of balls between x_0 and x if*

$$\begin{cases}
q_0 = x_0, \\
x \in \overline{B(q_N, \delta)}, \\
B(q_{j+1}, \delta) \subset B(q_j, 2\delta), \text{ for } j = 0, \dots, N-1, \\
B(q_j, 3\delta) \subset \Omega.
\end{cases}$$

Lemma 3.15. *Let x_0 and x in Ω. There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists a δ-sequence of balls between x_0 and x.*

Proof of Lemma 3.15. We refer to [Rob91] for a proof of this lemma. Let us just mention that in [Rob91], it is asserted that $x \in B(q_N, 2\delta)$, but looking carefully at the proof, we see that $x \in \overline{B(q_N, \delta)}$. □

We are now able to prove Proposition 3.6.

Proof of Proposition 3.6. Let us begin by proving inequality (3.10).

Let $x_0 \in \omega$ and $r_0 > 0$ be such that $B(x_0, r_0) \subset \omega$. For all $x \in \overline{\hat{\omega}}$, there exists, thanks to Lemma 3.15, a δ_x-sequence of balls $(B(q_j^x, \delta_x))_{j=0,\dots,N_x}$ between x_0 and x. Remark that we can assume that $\delta_x < r_0$, for all $x \in \overline{\hat{\omega}}$. The compact $\overline{\hat{\omega}}$ is included in $\bigcup_{x \in \overline{\hat{\omega}}} B(q_{N_x}^x, \delta_x)$, thus we can extract a finite sub-covering: there exists $\kappa \in \mathbb{N}^*$ and $(x_j)_{j=1,\dots,\kappa} \in \overline{\hat{\omega}}$ such that

$$\overline{\hat{\omega}} \subset \bigcup_{j=1,\dots,\kappa} B(q_{N_j}^j, \delta_j) \subset \bigcup_{j=1,\dots,\kappa} B(q_{N_j}^j, \delta), \tag{3.17}$$

where we have denoted for $j = 1, \dots, \kappa$, $N_j = N_{x_j}$, $\delta_j = \delta_{x_j}$, $q_i^j = q_i^{x_j}$ for $i = 0, \dots, N_j$ and where $\delta = \max_{j=1,\dots,\kappa} \delta_j$. Remark that we can assume that $N_j = N$ for all $j = 1, \dots, \kappa$ (if necessary, we consider several times the same ball).
To prove (3.10), it is sufficient to show that

$$\begin{cases}
\exists\, c, s > 0,\ \forall\, j = 1, \dots, \kappa,\ \forall\, i = 0, \dots, N-1,\ \forall\, \epsilon > 0,\ \forall\, (u,p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (1.1)}, \\
\|u\|_{H^1(B(q_{i+1}^j,\delta))} + \|p\|_{H^1(B(q_{i+1}^j,\delta))} \le \dfrac{c}{\epsilon}\left(\|u\|_{H^1(B(q_i^j,\delta))} + \|p\|_{H^1(B(q_i^j,\delta))}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right).
\end{cases} \tag{3.18}$$

Indeed, if this is the case, there exists $c, s > 0$ such that for all $j \in \{1, \dots, \kappa\}$, for all $\overline{\epsilon} > 0$ and $\tilde{\epsilon} > 0$ we have:

$$\begin{aligned}
\|u\|_{H^1(B(q_N^j,\delta))} + \|p\|_{H^1(B(q_N^j,\delta))} &\le \frac{c}{\overline{\epsilon}}(\|u\|_{H^1(B(q_{N-1}^j,\delta))} + \|p\|_{H^1(B(q_{N-1}^j,\delta))}) + \overline{\epsilon}^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \\
&\le \frac{c}{\overline{\epsilon}\tilde{\epsilon}}(\|u\|_{H^1(B(q_{N-2}^j,\delta))} + \|p\|_{H^1(B(q_{N-2}^j,\delta))}) + \left(\overline{\epsilon}^s + c\frac{\tilde{\epsilon}^s}{\overline{\epsilon}}\right)(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).
\end{aligned}$$

Let $\epsilon > 0$. Choosing successively $\tilde{\epsilon} = \frac{\overline{\epsilon}^{\frac{s+1}{s}}}{c^{\frac{1}{s}}}$ then $\overline{\epsilon} = \epsilon^{\frac{s}{2s+1}}$, we obtain that there exists $s > 0$ such that that:

$$\|u\|_{H^1(B(q_N^j,\delta))} + \|p\|_{H^1(B(q_N^j,\delta))} \le \frac{c}{\epsilon}(\|u\|_{H^1(B(q_{N-2}^j,\delta))} + \|p\|_{H^1(B(q_{N-2}^j,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

By iterating the process, we obtain the existence of $c, s > 0$ such that for all $j = 1, ..., \kappa$ and $(u,p) \in H^1(\Omega) \times H^1(\Omega)$ solution of (1.1):

$$\|u\|_{H^1(B(q_N^j,\delta))} + \|p\|_{H^1(B(q_N^j,\delta))} \leq \frac{c}{\epsilon}(\|u\|_{H^1(B(q_0^j,\delta))} + \|p\|_{H^1(B(q_0^j,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Note that $\delta < r_0$ and that, for all $j = 1, ..., \kappa$, $q_0^j = x_0$. By summing up the previous inequality for $j = 1, ..., \kappa$ and using (3.17), we obtain (3.10).

To prove (3.18), it suffices, thanks to the definition of a δ-sequence of balls, to prove the following inequality:

$$\left\{ \begin{array}{l} \exists\, c, s > 0,\ \forall\, j = 1, ..., \kappa,\ \forall\, i = 0, ..., N-1,\ \forall\, \epsilon > 0,\ \forall\, (u,p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (1.1)}, \\ \|u\|_{H^1(B(q_i^j,2\delta))} + \|p\|_{H^1(B(q_i^j,2\delta))} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(B(q_i^j,\delta))} + \|p\|_{H^1(B(q_i^j,\delta))}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right). \end{array} \right. \tag{3.19}$$

Let us emphasize that, thanks to Lemma 3.15, we can choose $\delta > 0$ in (3.17) small enough such that $B(q_i^j, 5\delta) \subset \Omega$ for all $j = 1, ..., \kappa$ and $i = 0, ..., N-1$ (it suffices to take $\delta \leq \frac{3\delta_0}{5}$).

Let $j \in \{1, ..., \kappa\}$ and $i \in \{0, ..., N\}$. We are going to apply Lemma 3.13 with $q = q_i^j$, $r_1 = \dfrac{\delta}{4}$, $r_2 = \dfrac{\delta}{2}$, $r_3 = 2\delta$, $r_4 = \dfrac{9\delta}{4}$, $r_5 = \dfrac{5\delta}{2}$.

We obtain that there exists $c > 0$, $h_1 > 0$, $c_1 = g\left(\frac{\delta}{4}\right) - g(2\delta) > 0$ and $c_2 = g(2\delta) - g\left(\frac{9\delta}{4}\right) > 0$ such that for all $h \in (0, h_1)$ and for all function $(u,p) \in H^1(B(q_i^j, \frac{5\delta}{2})) \times H^1(B(q_i^j, \frac{5\delta}{2}))$ solution of (3.13), we have

$$\begin{aligned} &\|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2},2\delta))} + \|p\|_{H^1(A_{q_i^j}(\frac{\delta}{2},2\delta))} \\ &\leq c\left(e^{c_1/h}\left(\|u\|_{H^1(B(q_i^j,\frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j,\frac{\delta}{2}))}\right) + e^{-c_2/h}\left(\|u\|_{H^1(B(q_i^j,\frac{5\delta}{2}))} + \|p\|_{H^1(B(q_i^j,\frac{5\delta}{2}))}\right)\right). \end{aligned} \tag{3.20}$$

Moreover, for all h small enough and for all functions $(u,p) \in H^1(\Omega) \times H^1(\Omega)$ solution of (1.1), the following inequality is obviously true:

$$\begin{aligned} &\|u\|_{H^1(B(q_i^j,\frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j,\frac{\delta}{2}))} \\ &\qquad \leq c\left(e^{c_1/h}(\|u\|_{H^1(B(q_i^j,\frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j,\frac{\delta}{2}))}) + e^{-c_2/h}(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)})\right). \end{aligned}$$

Since $B(q_i^j, \frac{5\delta}{2}) \subset \Omega$ and $B(q_i^j, \delta) \hookrightarrow B(q_i^j, \frac{\delta}{2})$, we get, summing up the two previous inequalities:

$$\begin{aligned} &\|u\|_{H^1(B(q_i^j,2\delta))} + \|p\|_{H^1(B(q_i^j,2\delta))} \\ &\qquad \leq c\left(e^{c_1/h}(\|u\|_{H^1(B(q_i^j,\delta))} + \|p\|_{H^1(B(q_i^j,\delta))}) + e^{-c_2/h}(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)})\right). \end{aligned} \tag{3.21}$$

Let us consider $\epsilon = e^{-c_1/h}$. We obtain that there exists $c > 0$, $s = \dfrac{c_2}{c_1} > 0$, such that for all $0 < \epsilon < \epsilon_1 = e^{-c_1/h_1}$, for all $(u,p) \in H^1(\Omega) \times H^1(\Omega)$ solution of (1.1), we have

$$\|u\|_{H^1(B(q_i^j,2\delta))} + \|p\|_{H^1(B(q_i^j,2\delta))} \leq c\left(\frac{1}{\epsilon}(\|u\|_{H^1(B(q_i^j,\delta))} + \|p\|_{H^1(B(q_i^j,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)})\right).$$

Observe that this inequality is still valid for $\epsilon \geq \epsilon_1$:

$$\begin{aligned} \|u\|_{H^1(B(q_i^j,2\delta))} + \|p\|_{H^1(B(q_i^j,2\delta))} &\leq \frac{\epsilon^s}{\epsilon^s}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right) \leq \frac{\epsilon^s}{\epsilon_1^s}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right) \\ &\leq c\left(\frac{1}{\epsilon}(\|u\|_{H^1(B(q_i^j,\delta))} + \|p\|_{H^1(B(q_i^j,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)})\right). \end{aligned}$$

This ends the proof of inequality (3.10).

Remark 3.16. *Let $\epsilon > 0$. Note that, from inequality (3.21), we could conclude the proof of Proposition 3.6 by using Lemma 3.4: there exists $C > 0$ such that,*

$$\begin{aligned}\|u\|_{H^1(B(q_i^j,2\delta))}+\|p\|_{H^1(B(q_i^j,2\delta))} &\leq C\left(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{H^1(B(q_i^j,\delta))}\right)^{\frac{c_2}{c_1+c_2}}\left(\|u\|_{H^1(\Omega)}+\|p\|_{H^1(\Omega)}\right)^{\frac{c_1}{c_1+c_2}}\\ &\leq C\left(\epsilon(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{H^1(B(q_i^j,\delta))})\right)^{\frac{c_2}{c_1+c_2}}\left(\epsilon^{\frac{c_2}{c_1}}(\|u\|_{H^1(\Omega)}+\|p\|_{H^1(\Omega)})\right)^{\frac{c_1}{c_1+c_2}}.\end{aligned}$$

Then, we conclude by using Young inequality.

Let us now prove inequality (3.11). In the same spirit as above, it is sufficient to prove the following inequality:

$$\left\{\begin{array}{l}\forall\, j=1,...,\kappa,\ \forall\, i=0,...,N-1,\ \exists\, c,s>0,\ \forall\,\epsilon>0,\ \forall\,(u,p)\in H^1(\Omega)\times L^2(\Omega)\text{ solution of (1.1)},\\ \|u\|_{H^1(B(q_i^j,2\delta))}+\|p\|_{L^2(B(q_i^j,2\delta))}\leq \dfrac{c}{\epsilon}\left(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{L^2(B(q_i^j,\delta))}\right)+\epsilon^s\left(\|u\|_{H^1(\Omega)}+\|p\|_{L^2(\Omega)}\right).\end{array}\right.$$

Note that, since (u,p) is solution of (1.1), (u,p) belongs to $H^2_{loc}(\Omega)\times H^1_{loc}(\Omega)$ (we refer to [BF06] for a proof of this result). Considering again inequality (3.20), we will use Caccioppoli inequality to get rid of the L^2-norm of ∇p.

Lemma 3.17 (Caccioppoli inequality)**.** *Let v be a weak solution of $\Delta v = 0$ in $\Omega\subset\mathbb{R}^d$. Then, there exists $C>0$ such that for all $x_0\in\Omega$ and $0<\rho<R<d(x_0,\partial\Omega)$, we have*

$$\int_{B(x_0,\rho)}|\nabla v|^2\leq\frac{C}{(R-\rho)^2}\int_{B(x_0,R)}|v|^2.$$

We refer to [GS85] for more details about Caccioppoli inequality. Thus, thanks to Caccioppoli inequality, there exists $c>0$ such that:

$$\|\nabla p\|_{L^2(B(q_i^j,\frac{\delta}{2}))}\leq c\|p\|_{L^2(B(q_i^j,\delta))}\quad\text{and}\quad\|\nabla p\|_{L^2(B(q_i^j,\frac{5\delta}{2}))}\leq c\|p\|_{L^2(B(q_i^j,5\delta))}.$$

By coupling this with inequality (3.20), we obtain that:

$$\begin{aligned}&\|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2},2\delta))}+\|p\|_{L^2(A_{q_i^j}(\frac{\delta}{2},2\delta))}\leq\|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2},2\delta))}+\|p\|_{H^1(A_{q_i^j}(\frac{\delta}{2},2\delta))}\\ &\leq c\left(e^{c_1/h}(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{L^2(B(q_i^j,\delta))})+e^{-c_2/h}(\|u\|_{H^1(B(q_i^j,5\delta))}+\|p\|_{L^2(B(q_i^j,5\delta))})\right).\end{aligned}\tag{3.22}$$

Moreover, for all h small enough and for all functions $(u,p)\in H^1(B(q_i^j,5\delta))\times L^2(B(q_i^j,5\delta))$, the following inequality is obviously true:

$$\begin{aligned}\|u\|_{H^1(B(q_i^j,\frac{\delta}{2}))}+\|p\|_{L^2(B(q_i^j,\frac{\delta}{2}))}&\leq ce^{c_1/h}(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{L^2(B(q_i^j,\delta))})\\ &\quad+ce^{-c_2/h}(\|u\|_{H^1(B(q_i^j,5\delta))}+\|p\|_{L^2(B(q_i^j,5\delta))}).\end{aligned}\tag{3.23}$$

By summing up inequalities (3.22) and (3.23), we get:

$$\begin{aligned}&\|u\|_{H^1(B(q_i^j,2\delta))}+\|p\|_{L^2(B(q_i^j,2\delta))}\\ &\leq c\left(e^{c_1/h}(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{L^2(B(q_i^j,\delta))})+e^{-c_2/h}(\|u\|_{H^1(B(q_i^j,5\delta))}+\|p\|_{L^2(B(q_i^j,5\delta))})\right).\end{aligned}\tag{3.24}$$

Since $B(q_i^j,5\delta)\subset\Omega$, this leads to:

$$\|u\|_{H^1(B(q_i^j,2\delta))}+\|p\|_{L^2(B(q_i^j,2\delta))}\leq c\left(e^{c_1/h}(\|u\|_{H^1(B(q_i^j,\delta))}+\|p\|_{L^2(B(q_i^j,\delta))})+e^{-c_2/h}(\|u\|_{H^1(\Omega)}+\|p\|_{L^2(\Omega)})\right).$$

In the same way that we concluded the proof of inequality (3.10), we can finish the proof of inequality (3.11) by considering $\epsilon=e^{-c_1/h}$ or by using Lemma 3.4 (see remark 3.16). □

Remark 3.18. *Let us notice that inequality* (3.10) *of Proposition 3.6 implies that:*

$$\left\{ \begin{array}{l} \forall\,\beta > 0,\ \exists\, c > 0, \forall\, \epsilon > 0, \forall\,(u,p) \in H^1(\Omega)\times H^1(\Omega) \text{ solution of } (1.1), \\ \|u\|_{H^1(\hat\omega)} + \|p\|_{H^1(\hat\omega)} \le e^{\frac{c}{\epsilon}} \left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^{\beta}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right). \end{array}\right.$$

Indeed, we have for all $\beta > 0$*:*

$$\left\{ \begin{array}{l} \exists\, c, s > 0, \forall\, \epsilon > 0, \forall\,(u,p) \in H^1(\Omega)\times H^1(\Omega) \text{ solution of } (1.1), \\ \|u\|_{H^1(\hat\omega)} + \|p\|_{H^1(\hat\omega)} \le \left(\dfrac{c}{\epsilon}\right)^{\frac{\beta}{s}} \left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^{\beta}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right), \end{array}\right.$$

and $\left(\dfrac{c}{\epsilon}\right)^{\frac{\beta}{s}} = e^{\frac{\beta}{s}\ln(\frac{c}{\epsilon})} \le e^{\frac{c}{\epsilon}}$*, since for all* $x > 0$, $\ln(x) \le x$.

Let $0 < \nu \le \frac{1}{2}$*. Note that we will use this remark for* $\beta \in \left(0, \dfrac{1}{2} + \nu\right)$ *in the proof of Theorem 3.2.*

Remark 3.19. *The proof of Proposition 3.6 contains all the tools needed to prove an interesting result which is, in the case of the Stokes system, a three balls inequality involving the velocity in* H^1 *norm and the pressure in* L^2 *norm. Let us be more precise. Let* $\delta > 0$ *and* $q \in \mathbb{R}^d$*. One can prove that there exist* $c > 0$, $\alpha > 0$ *such that for all functions* $(u,p) \in H^1(B(q,5\delta)) \times L^2(B(q,5\delta))$ *solution of:*

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } B(q,5\delta), \\ \mathit{div}\ u & = & 0, & \text{in } B(q,5\delta), \end{array}\right.$$

the following inequality is satisfied:

$$\begin{aligned} &\|u\|_{H^1(B(q,2\delta))} + \|p\|_{L^2(B(q,2\delta))} \\ &\qquad \le c\left(\|u\|_{H^1(B(q,\delta))} + \|p\|_{L^2(B(q,\delta))}\right)^{\alpha}\left(\|u\|_{H^1(B(q,5\delta))} + \|p\|_{L^2(B(q,5\delta))}\right)^{1-\alpha}, \end{aligned}$$

with $\alpha = \dfrac{g(2\delta) - g(\frac{9\delta}{4})}{g(\frac{\delta}{4}) - g(\frac{9\delta}{4})}$ *and* $g(r) = e^{-\lambda r^2}$ *with* λ *large enough. To do so, one can prove inequality* (3.24) *with* $q_i^j = q$*. Then, it suffices to apply Lemma 3.4* $D = \|u\|_{H^1(B(q,2\delta))} + \|p\|_{L^2(B(q,2\delta))}$, $A = \|u\|_{H^1(B(q,\delta))} + \|p\|_{L^2(B(q,\delta))}$ *and* $B = \|u\|_{H^1(B(q,5\delta))} + \|p\|_{L^2(B(q,5\delta))}$*, remembering that* $c_1 = g(\frac{\delta}{4}) - g(2\delta)$ *and* $c_2 = g(2\delta) - g(\frac{9\delta}{4})$.

We refer to [BD10] for a three balls inequality for the Laplacian. Note that in [LUW10], C.-H. Lin, G. Uhlmann and J.-N. Wang have obtained an optimal three balls inequality for the Stokes system involving the velocity in L^2 *norm. From this inequality, they derive an upper bound on the vanishing order of any non trivial solution u to the Stokes system.*

3.3 Estimates near the boundary: proof of Propositions 3.8 and 3.9

To prove Propositions 3.8 and 3.9, we are going to apply the local Carleman inequality near the boundary given by Proposition 2.4. To do this, we must locally go back to the half-plane: we use the system of geodesic normal coordinates. In the system of geodesic normal coordinates, the Laplace operator is transported to an operator satisfying the assumptions of Proposition 2.4 (see [Hör85]).

We first state a lemma which will be used in the proof of Proposition 3.8.

Lemma 3.20. *Let* $0 < \nu \le \frac{1}{2}$, $0 < r_0 \le R_0$, $K = \{x \in \mathbb{R}^d_+ / |x| \le R_0\}$, $(f,g) \in L^2(K) \times L^2(K)$, $B \in GL_d(\mathcal{C}^\infty(K))$ *and* P *be a second-order differential operator whose coefficients are* $\mathcal{C}^\infty$ *in a neighborhood of* K*, defined by* $P(x,\partial_x) = -\partial_{x_d}^2 + R(x, \frac{1}{i}\partial_{x'})$*. Let us denote by* $r(x,\xi')$ *the principal symbol of* R*. We assume that* $r(x,\xi') \in \mathbb{R}$ *and that there exists a constant* $c > 0$ *such that* $(x,\xi') \in K \times \mathbb{R}^{d-1}$*, we have* $r(x,\xi') \ge c|\xi'|^2$.

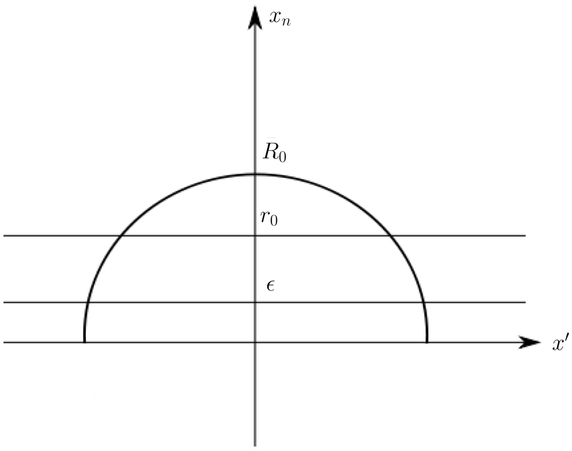

Figure 1: Considered sets in the proof of Proposition 3.8 (in dimension 2).

We denote by $K(r,r') = \{x \in K / r < x_d < r'\}$, for $0 < r < r' < R_0$. Then, for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$ such that for all $\epsilon > 0$, the following inequality holds

$$\begin{aligned}&\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))}\\ &\le e^{\frac{c}{\epsilon}} \left(\|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}\right) + \epsilon^\beta \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right).\end{aligned}$$

for all $(v,q) \in H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K) \times H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K)$ solution of

$$\left\{\begin{array}{rcll} -Pv + B\nabla q &=& f, & \text{in } K,\\ Pq &=& g, & \text{in } K.\end{array}\right. \tag{3.25}$$

Proof of Lemma 3.20. Let $0 < \epsilon < \epsilon_0 < r_0 < R_0$, we denote by $U = K(0,r_0)$ and $U_\epsilon = K(\epsilon, r_0)$. We are going to apply Proposition 2.4 with $\phi(x) = e^{\lambda x_d}$, for λ large enough. Let $\chi \in \mathcal{C}^\infty(K)$ be a function equal to zero in K^c, such that $\chi = 1$ in U, $0 \le \chi \le 1$ in $K\setminus U$. Notice that:

$$U_\epsilon \subset U \subset K.$$

We apply successively Carleman inequality stated in Proposition 2.4 to χv and χq, taking into account Lemma 2.8: $\exists\, c > 0, h_1 > 0$, $\forall\, 0 < h < h_1$, $\forall\, (v,q) \in H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K) \times H^{\frac{3}{2}+\nu}_{0,\partial K\setminus\Sigma}(K)$ solution of (3.25):

$$\begin{aligned}&\int_U |v(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_U |\nabla v(x)|^2 e^{2\phi(x)/h} dx\\ &\qquad\le ch^3 \int_K |\chi P v(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K\setminus U} |[P,\chi] v(x)|^2 e^{2\phi(x)/h} dx\\ &\qquad + c\int_{\mathbb{R}^{d-1}} \left(|\chi v(x',0)|^2 + |h\partial_{x'}(\chi v)(x',0)|^2 + |h\partial_{x_d}(\chi v)(x',0)|^2\right) e^{2\phi(x',0)/h} dx',\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}&\int_U |q(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\chi \nabla q(x)|^2 e^{2\phi(x)/h} dx\\ &\le ch^3 \int_K |\chi P q(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K\setminus U} |[P,\chi] q(x)|^2 e^{2\phi(x)/h} dx + ch^2 \int_{K\setminus U} |q(x)|^2 e^{2\phi(x)/h} dx\\ &\qquad + c\int_{\mathbb{R}^{d-1}} \left(|\chi q(x',0)|^2 + |h\partial_{x'}(\chi q)(x',0)|^2 + |h\partial_{x_d}(\chi q)(x',0)|^2\right) e^{2\phi(x',0)/h} dx'.\end{aligned} \tag{3.27}$$

Since (v,q) satisfies (3.25), we can respectively replace χPv and χPq in the two previous inequalities by $-\chi B\nabla q+\chi f$ and χg. Note that there exists a constant $c>0$ such that $|B\nabla q|^2\le c|\nabla q|^2$. Then, by summing up inequalities (3.26) and (3.27), the term $ch^3\int_K |\chi\nabla q(x)|^2 e^{2\phi(x)/h}dx$ which appears in the right hand-side will be absorbed by the term $h^2\int_K |\chi\nabla q(x)|^2e^{2\phi(x)/h}dx$ which is in the left hand-side, for h small enough. By dividing by h^2, we obtain, for h small enough:

$$\begin{aligned}
&\int_{U_\epsilon}\left(|v(x)|^2+|q(x)|^2\right)e^{2\phi(x)/h}dx+\int_{U_\epsilon}\left(|\nabla v(x)|^2+|\nabla q(x)|^2\right)e^{2\phi(x)/h}dx\\
&\qquad\le c\int_K |f(x)|^2e^{2\phi(x)/h}dx+c\int_K|g(x)|^2e^{2\phi(x)/h}dx+c\int_{K\setminus U}|q(x)|^2e^{2\phi(x)/h}dx\\
&+c\int_{K\setminus U}\left(|[P,\chi]v(x)|^2+|[P,\chi]q(x)|^2\right)e^{2\phi(x)/h}dx+\frac{c}{h^2}\int_{\mathbb{R}^{d-1}}\left(|\chi v(x',0)|^2+|\chi q(x',0)|^2\right)e^{2\phi(x',0)/h}dx'\\
&+\frac{c}{h^2}\int_{\mathbb{R}^{d-1}}\left(|\partial_{x'}(\chi v)(x',0)|^2+|\partial_{x'}(\chi q)(x',0)|^2+|\partial_{x_d}(\chi v)(x',0)|^2+|\partial_{x_d}(\chi q)(x',0)|^2\right)e^{2\phi(x',0)/h}dx'.
\end{aligned}$$

By replacing $\phi(x)$ by $e^{\lambda x_d}$, using Lemma 2.8 and thanks to the trace inequality, the previous inequality becomes

$$\begin{aligned}
&e^{\frac{e^{\lambda\epsilon}}{h}}\left(\|v\|_{H^1(K(\epsilon,r_0))}+\|q\|_{H^1(K(\epsilon,r_0))}\right)\\
&\qquad\le ce^{\frac{e^{\lambda R_0}}{h}}\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)\\
&\qquad\qquad+\frac{c}{h}e^{\frac{1}{h}}\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right).
\end{aligned}$$

Remark that for all $\epsilon\ge 0$, $-e^{\lambda\epsilon}+1\le-\epsilon$ as long as λ is large enough. Thus:

$$\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\le ce^{\frac{c}{h}}\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)\\
&\qquad\qquad+\frac{c}{h}e^{-\frac{\epsilon}{h}}\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right).
\end{aligned}$$

Moreover, for all $\epsilon\ge 0$, $\dfrac{1}{h}\le\dfrac{2}{\epsilon}e^{\frac{\epsilon}{2h}}$, which implies:

$$\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\le ce^{\frac{c}{h}}\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)\\
&\qquad\qquad+\frac{c}{\epsilon}e^{-\frac{\epsilon}{2h}}\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right).
\end{aligned}$$

According to Lemma 3.4, we obtain:

$$\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\le c\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)^{\frac{\epsilon}{\epsilon+c}}\left(\frac{1}{\epsilon}\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right)\right)^{1-\frac{\epsilon}{\epsilon+c}}.
\end{aligned}$$

Let $s>0$ and $\mu>1$. The previous estimate can be rewritten as:

$$\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\quad\le c\left(\epsilon^{-\frac{c}{\epsilon}(s+1)}\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)\right)^{\frac{\epsilon}{\epsilon+c}}\times\\
&\qquad\qquad\left(\epsilon^s\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right)\right)^{1-\frac{\epsilon}{\epsilon+c}}\\
&\quad\le c\epsilon^{-\frac{c(s+1)}{\epsilon}}\left(\|v\|_{H^1(K(r_0,R_0))}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)}+\|g\|_{L^2(K)}\right)\\
&\qquad\qquad+\epsilon^s\left(\|v\|_{H^{\frac32+\nu}(K)}+\|q\|_{H^{\frac32+\nu}(K)}\right).
\end{aligned}$$

But $\epsilon^{-\frac{c(s+1)}{\epsilon}} = \exp\left(\frac{c}{\epsilon}(s+1)\ln\left(\frac{1}{\epsilon}\right)\right) \leq \exp\left(\frac{c(s+1)}{(\mu-1)\epsilon^{\mu}}\right)$ since $(\mu-1)\ln\left(\frac{1}{\epsilon}\right) \leq \frac{1}{\epsilon^{\mu-1}} \leq \frac{1}{\epsilon^{\mu}}$ for ϵ small enough. Finally, $\forall\, s>0$, $\forall\, \mu>1$, $\exists\, c>0$, $\forall\, 0<\epsilon<\epsilon_0$,

$$\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))} + \|q\|_{H^1(K(\epsilon,r_0))} \\
&\qquad \leq c e^{\frac{c}{\epsilon^{\mu}}} \left(\|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}\right) \\
&\qquad\qquad + \epsilon^{s}\left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right), \qquad (3.28)
\end{aligned}$$

for all (v,q) solution of (3.25).

It remains to estimate $\|v\|_{H^1(K(0,\epsilon))} + \|q\|_{H^1(K(0,\epsilon))}$ uniformly in ϵ. This is a consequence of Hardy inequality:

Lemma 3.21 (Hardy inequality)**.** *Let $0<\tau<\frac{1}{2}$. There exists $c>0$ such that for all $h \in H^{\tau}(\mathbb{R}^d_+)$, we have*

$$\left\|\frac{h}{x_d^{\tau}}\right\|_{L^2(\mathbb{R}^d_+)} \leq c\|h\|_{H^{\tau}(\mathbb{R}^d_+)}.$$

We refer to [DL90] for a proof of Lemma 3.21.

We extend v and q by zero in $\mathbb{R}^d_+ \backslash K$. Note that these extensions, denoted respectively by $\tilde{v}$ and $\tilde{q}$, belong to $H^{\frac{3}{2}+\nu}(\mathbb{R}^d_+)$ (see [eEM68]). Let $\tilde{\chi}$ be a function which belongs to $\mathcal{C}^{\infty}_c(\{(x',x_d) \in \mathbb{R}^d_+ / x_d < r_0\})$ such that $\tilde{\chi}=1$ on $K(0,\epsilon)$ and $0 \leq \tilde{\chi} \leq 1$ elsewhere. The functions $\tilde{\chi}\tilde{v}$ and $\tilde{\chi}\tilde{q}$ belong to $H^{\frac{3}{2}+\nu}(\mathbb{R}^d_+)$, therefore as a result of Hardy inequality we have that for all $0<\tau<\frac{1}{2}$, that there exists $c>0$, such that

$$\left\|\frac{v}{x_d^{\tau}}\right\|_{L^2(K(0,\epsilon))} \leq \left\|\frac{\tilde{\chi}\tilde{v}}{x_d^{\tau}}\right\|_{L^2(\mathbb{R}^d_+)} \leq c\|\tilde{\chi}\tilde{v}\|_{H^{\tau}(\mathbb{R}^d_+)}.$$

Since $\tilde{\chi}\tilde{v}=0$ in $\left(\mathbb{R}^d_+\backslash K\right)\bigcup K(r_0,R_0)$, we obtain

$$\left\|\frac{v}{x_d^{\tau}}\right\|_{L^2(K(0,\epsilon))} \leq c\|v\|_{H^{\tau}(K(0,r_0))} \leq c\|v\|_{H^{\frac{1}{2}}(K(0,r_0))}.$$

Consequently, for all $\tau \in \left(0,\frac{1}{2}\right)$, there exists $c>0$, such that for all $\alpha>0$,

$$\begin{aligned}
\|v\|_{L^2(K(0,\epsilon))} \leq c\epsilon^{\tau}\|v\|_{H^{\frac{1}{2}}(K(0,r_0))} &\leq c\epsilon^{\tau}\|v\|^{\frac{1}{2}}_{H^1(K)}\|v\|^{\frac{1}{2}}_{L^2(K(0,r_0))} \\
&\leq c\left(\frac{\epsilon^{2\tau}}{\alpha}\|v\|_{H^1(K)} + \alpha\|v\|_{L^2(K(0,r_0))}\right),
\end{aligned}$$

where we used an interpolation inequality and Young inequality. In the same way, we have for ∇v:

$$\begin{aligned}
\|\nabla v\|_{L^2(K(0,\epsilon))} \leq c\epsilon^{\tau}\|\nabla v\|_{H^{\frac{1}{2}}(K(0,r_0))} &\leq c\epsilon^{\tau}\|\nabla v\|^{\frac{1}{1+2\nu}}_{H^{\frac{1}{2}+\nu}(K)}\|\nabla v\|^{\frac{2\nu}{1+2\nu}}_{L^2(K(0,r_0))} \\
&\leq c\left(\epsilon^{\tau(1+2\nu)}\frac{1}{\alpha^{2\nu}}\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \alpha\|v\|_{H^1(K(0,r_0))}\right).
\end{aligned}$$

To summarize, for all $\beta \in \left(0,\dfrac{1}{2}+\nu\right)$, there exists $c>0$, such that for all $0<\alpha<1$,

$$\|v\|_{H^1(K(0,\epsilon))} \leq c\left(\frac{\epsilon^{\beta}}{\alpha}\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \alpha\|v\|_{H^1(K(0,r_0))}\right).$$

The same inequality also holds for q. Thus, for all $\beta \in \left(0,\dfrac{1}{2}+\nu\right)$, there exists $c>0$, such that for all $0<\alpha<1$,

$$\begin{aligned}
&\|v\|_{H^1(K(0,\epsilon))} + \|q\|_{H^1(K(0,\epsilon))} \\
&\qquad \leq c\left(\frac{\epsilon^{\beta}}{\alpha}\left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right) + \alpha\left(\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))}\right)\right). \qquad (3.29)
\end{aligned}$$

We can choose α small enough such that by combining (3.28) and (3.29) we have the following assertion: $\forall \beta \in \left(0, \dfrac{1}{2}+\nu\right)$, $\forall\, \mu > 1$, $\exists\, c > 0$, $\forall\, 0 < \epsilon < \epsilon_0$,

$$\begin{aligned}
&\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\
&\qquad\qquad \le c e^{\frac{c}{\epsilon^\mu}} \left(\|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + c\epsilon^\beta \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right).
\end{aligned}$$

By a change of variables, we obtain: for all $\beta \in \left(0, \dfrac{1}{2}+\nu\right)$, there exists $c > 0$ such that for all $0 < \epsilon < \tilde{\epsilon}_0$,

$$\begin{aligned}
&\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\
&\qquad\qquad \le e^{\frac{c}{\epsilon}} \left(\|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \epsilon^\beta \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right).
\end{aligned} \tag{3.30}$$

This last inequality remains true for $\epsilon \ge \tilde{\epsilon}_0$, since $H^{\frac{3}{2}+\nu}(K) \hookrightarrow H^1(K(0,r_0))$:

$$\begin{aligned}
&\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\
&\qquad \le c\frac{\epsilon^\beta}{\epsilon^\beta} \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right) \le c\frac{\epsilon^\beta}{(\tilde{\epsilon}_0)^\beta} \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right) \\
&\qquad\qquad \le e^{\frac{c}{\epsilon}} \left(\|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + c\epsilon^\beta \left(\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}\right),
\end{aligned}$$

which ends the proof of Lemma 3.20. $\square$

Let us now prove Proposition 3.8.

Proof of Proposition 3.8. We are first going to prove that there exists an open neighborhood $\hat{\omega}$ of x_0 and two relatively compact open sets $\tilde{\omega}_1 \subset \Omega$ and $\tilde{\omega}_2 \subset \Omega$ such that:

$$\forall\, \beta \in \left(0, \frac{1}{2}+\nu\right), \exists\, c > 0, \forall\, \epsilon > 0, \forall\, (u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

$$\begin{aligned}
&\|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \le e^{\frac{c}{\epsilon}} \left(\|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)} + \|p\|_{H^1(\tilde{\omega}_2)}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \epsilon^\beta \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}\right).
\end{aligned} \tag{3.31}$$

Then, to pass from (3.31) to (3.12) to obtain the estimate for any ω, it is sufficient to apply inequality (3.10) of Proposition 3.6.

Let $\mathcal{V}$ be a neighborhood of x_0 such that $\Omega \cap \mathcal{V} = \{(x', x_d) \in \mathcal{V} / x_d > \sigma(x')\}$ with $\sigma \in \mathcal{C}^\infty$. By using the normal geodesic coordinates, it is possible to straighten locally in a neighborhood $\mathcal{V}$ of x_0 simultaneously the Laplace operator and the boundary. Restricting if necessary the open set $\mathcal{V}$, we can assume that there exists a neighborhood $\tilde{\mathcal{V}} \subset \mathcal{V}$ of x_0, a surface S such that $S \cap \tilde{\mathcal{V}} = \partial\Omega \cap \tilde{\mathcal{V}}$ and S is deformed inwardly the open set Ω in $\mathcal{V}\backslash\tilde{\mathcal{V}}$ (this means that there exists $s \in \mathcal{C}^\infty$ such that $S = \{(x', x_d) \in \mathcal{V} / x_d = s(x')\}$ with $s = \sigma$ in $\tilde{\mathcal{V}}$ and $s > \sigma$ in $\mathcal{V}\backslash\tilde{\mathcal{V}}$) and a diffeomorphism, denoted ψ, which straightens both S and the Laplace operator. Let us denote by $\tilde{\Omega} = \{(x', x_d) \in \mathcal{V} / x_d > s(x')\}$. More precisely, ψ is such that

1. $\psi(x_0) = 0$,
2. there exists $R_0 > 0$ such that $\psi(\tilde{\Omega}) = \{x \in \mathbb{R}^d_+ / |x| < R_0\}$,

3. $\psi(S) = \{(x', x_d) \in \mathbb{R}^d / x_d = 0 \text{ and } |x| < R_0\}$,

4. the transported operator P satisfies the assumptions of Proposition 2.4 in $K = \{x \in \mathbb{R}^d_+ / |x| \leq R_0\}$.

Note that, by construction, there exists $0 < r_3 < R_0$, such that $\psi^{-1}(\{x \in K/r_3 < |x|\})$ is a relatively compact open set of Ω. Let $\xi \in \mathcal{C}^\infty_c(\overline{K})$ be such that $\xi = 1$ in $\{x \in \mathbb{R}^d_+/|x| \leq r_3\}$ and $0 \leq \xi \leq 1$ elsewhere. Let us denote by $\varrho = \xi \circ \psi$. Note that since $(v, q) = (\varrho u, \varrho p)$ is solution in $\tilde{\Omega} \cap \mathcal{V}$ of

$$\left\{ \begin{array}{rcl} -\Delta v + \nabla q & = & f, \\ \Delta q & = & g, \end{array} \right.$$

with $f = -u\Delta\varrho - 2\nabla u \nabla \varrho + \nabla \varrho p$ and $g = \Delta \varrho p + 2\nabla\varrho \cdot \nabla p$, then $(w, \pi) = ((\varrho u) \circ \psi^{-1}, (\varrho p) \circ \psi^{-1})$ is solution in K of

$$\left\{ \begin{array}{rcl} -Pw + (\nabla\psi)^T \nabla \pi & = & f \circ \psi^{-1}, \\ P\pi & = & g \circ \psi^{-1}. \end{array} \right. \tag{3.32}$$

We apply Lemma 3.20 to (w, π). We obtain that for all $\beta \in \left(0, \dfrac{1}{2} + \nu\right)$, there exists $c > 0$ such that for all $\epsilon > 0$, for all $(w, \pi) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (3.32),

$$\begin{aligned} &\|w\|_{H^1(K(0,r_0)\cap B(0,r_3))} + \|\pi\|_{H^1(K(0,r_0)\cap B(0,r_3))} \\ &\qquad \leq e^{\frac{c}{\epsilon}} \left(\|w\|_{H^1(K(r_0,R_0))} + \|\pi\|_{H^1(K(r_0,R_0))} + \|f \circ \psi^{-1}\|_{L^2(K)} + \|g \circ \psi^{-1}\|_{L^2(K)} \right) \\ &\qquad\qquad + \epsilon^\beta \left(\|w\|_{H^{\frac{3}{2}+\nu}(K)} + \|\pi\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned}$$

In other words, there exists an open neighborhood $\hat{\omega}$ of x_0 and a relatively compact open set $\tilde{\omega}_1 \subset \Omega$ such that:

$$\forall\, \beta \in \left(0, \frac{1}{2} + \nu\right), \exists\, c > 0, \forall\, \epsilon > 0, \forall\, (u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

$$\begin{aligned} \|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} &\leq e^{\frac{c}{\epsilon}} \left(\|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|f\|_{L^2(\tilde{\Omega}\cap\mathcal{V})} + \|g\|_{L^2(\tilde{\Omega}\cap\mathcal{V})} \right) \\ &\qquad + \epsilon^\beta \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \end{aligned}$$

To conclude, let us remark that since $\xi = 1$ in $\{x \in \mathbb{R}^d_+/|x| \leq r_3\}$, $\text{supp}(\nabla\xi) \subset \{x \in K/r_3 < |x|\}$ and then $\text{supp}(\nabla\varrho) \subset \psi^{-1}(\{x \in K/r_3 < |x|\})$ which is a relatively compact open set of Ω. Then, remembering the definition of f and g, we obtain that there exists a relatively compact open set $\tilde{\omega}_2 \subset \Omega$ such that:

$$\forall\, \beta \in \left(0, \frac{1}{2} + \nu\right), \exists\, c > 0, \forall\, \epsilon > 0, \forall\, (u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

$$\begin{aligned} \|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} &\leq e^{\frac{c}{\epsilon}} \left(\|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)} + \|p\|_{H^1(\tilde{\omega}_2)} \right) \\ &\qquad + \epsilon^\beta \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \end{aligned}$$

We obtain (3.31) which ends the proof. □

Let us end this subsection with the proof of Proposition 3.9.

Proof of Proposition 3.9. Let $x_0 \in \Gamma$. We are going to prove that there exists a neighborhood ω of x_0 such that:

$$\exists\, c, s > 0, \forall\, \epsilon > 0, \forall\, (u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

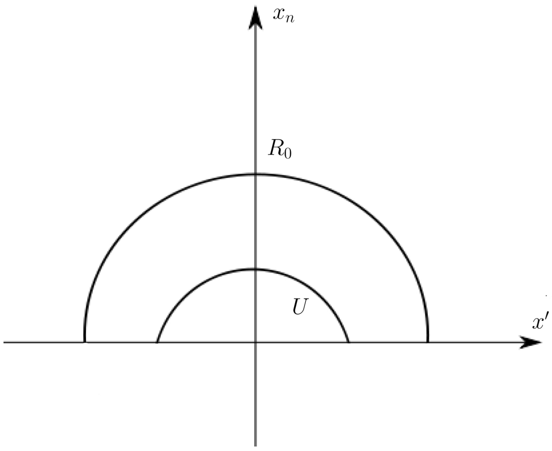

Figure 2: Considered sets in the proof of Proposition 3.9 (in dimension 2).

$$\|u\|_{H^1(\omega\cap\Omega)} + \|p\|_{H^1(\omega\cap\Omega)} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right). \quad (3.33)$$

This inequality implies Proposition 3.9 thanks to inequality (3.10) of Proposition 3.6.

Near the boundary, in a neighborhood of x_0, we go back to the half-plane thanks to geodesic normal coordinates: let ψ and $\mathcal{V}$ be such that

1. $\psi(x_0) = 0$,
2. there exists $R_0 > 0$ such that $\psi(\Omega \cap \mathcal{V}) = \{x \in \mathbb{R}^d_+ / |x| < R_0\}$,
3. $\psi(\partial\Omega \cap \mathcal{V}) = \{(x', x_d) \in \mathbb{R}^d / x_d = 0 \text{ and } |x| < R_0\}$,
4. the transported operator P satisfies the assumptions of Proposition 2.4 in $K = \{x \in \mathbb{R}^d_+ / |x| \leq R_0\}$.

We can always assume that $\mathcal{V}$ is small enough to have $\partial\Omega \cap \mathcal{V} \subset \Gamma$. In the sequel, we denote by $\Sigma = \psi(\partial\Omega \cap \mathcal{V}) \subset \mathbb{R}^{d-1}$. Let us denote by $(v, q) = (u \circ \psi^{-1}, p \circ \psi^{-1})$. Note that (v, q) is solution in K of

$$\begin{cases} -Pv + (\nabla\psi)^T \nabla q &= 0, \\ Pq &= 0. \end{cases} \quad (3.34)$$

We are going to prove that there exists a neighborhood θ of 0 such that:

$$\exists\, c, s > 0, \forall\, \epsilon > 0, \forall\, (v, q) \in H^{\frac{3}{2}+\nu}(K) \times H^{\frac{3}{2}+\nu}(K) \text{ solution of (3.34)},$$
$$\|v\|_{H^1(K\cap\theta)} + \|q\|_{H^1(K\cap\theta)} \leq \frac{c}{\epsilon}\left(\|v\|_{H^1(\Sigma)} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_d} v\|_{L^2(\Sigma)} + \|\partial_{x_d} q\|_{L^2(\Sigma)}\right) + \epsilon^s(\|v\|_{H^1(K)} + \|q\|_{H^1(K)}).$$

To obtain this inequality, we apply Proposition 2.4 with $\phi(x) = e^{-\lambda(x_d + |x|^2)}$ and λ large enough. Let $U = \{x \in K / x_d + |x|^2 \leq r_0\}$ with r_0 small enough (see Figure 2) and $\chi \in \mathcal{C}^\infty_c(\overline{K})$ be such that $\chi = 1$ on U, $0 \leq \chi \leq 1$ in $K \backslash U$. By successive applications of Proposition 2.4 to χv and to χq, we obtain (in the same way as in the proof of Proposition 3.8):

$$\exists\, c>0,\ h_1>0,\ \forall\, 0<h<h_1,\ \forall\,(v,q)\in H^{\frac{3}{2}+\nu}(K)\times H^{\frac{3}{2}+\nu}(K) \text{ satisfying (3.34)}$$

$$\int_U (|v(x)|^2+|q(x)|^2)e^{2\phi(x)/h}dx+h^2\int_U(|\nabla v(x)|^2+|\nabla q(x)|^2)e^{2\phi(x)/h}dx\le ch^3\int_{K\setminus U}|\nabla q(x)|^2e^{2\phi(x)/h}dx$$
$$+ch^2\int_{K\setminus U}|q(x)|^2e^{2\phi(x)/h}dx+ch^3\int_{K\setminus U}(|[P,\chi]v(x)|^2+|[P,\chi]q(x)|^2)e^{2\phi(x)/h}dx$$
$$+c\int_{\mathbb{R}^{d-1}}(|h\partial_{x'}(\chi v)(x',0)|^2+|h\partial_{x'}(\chi q)(x',0)|^2+|h\partial_{x_d}(\chi v)(x',0)|^2+|h\partial_{x_d}(\chi q)(x',0)|^2)e^{2\phi(x',0)/h}dx'$$
$$+c\int_{\mathbb{R}^{d-1}}(|\chi v(x',0)|^2+|\chi q(x',0)|^2)e^{2\phi(x',0)/h}dx'.$$

We denote by $R(r,r')=\{x\in K/r<x_d+|x|^2<r'\}$. The previous inequality becomes, with $0<z_1<r_0<z_2<R_0$:

$$e^{\frac{e^{-\lambda z_1}}{h}}(\|v\|_{H^1(R(0,z_1))}+\|q\|_{H^1(R(0,z_1))})\le ce^{\frac{e^{-\lambda z_2}}{h}}(\|v\|_{H^1(R(z_2,R_0))}+\|q\|_{H^1(R(z_2,R_0))})$$
$$+ce^{\frac{1}{h}}(\|v\|_{H^1(\Sigma)}+\|\partial_{x_d}v\|_{L^2(\Sigma)}+\|q\|_{H^1(\Sigma)}+\|\partial_{x_d}q\|_{L^2(\Sigma)}).$$

Accordingly:

$$\exists\, c,h_1>0,\ \forall\, 0<h<h_1,\ \forall\,(v,q)\in H^{\frac{3}{2}+\nu}(\Omega)\times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.34)},$$
$$\|v\|_{H^1(R(0,z_1))}+\|q\|_{H^1(R(0,z_1))}\le ce^{-\frac{1}{h}}(\|v\|_{H^1(K)}+\|q\|_{H^1(K)})$$
$$+ce^{\frac{c}{h}}(\|v\|_{H^1(\Sigma)}+\|\partial_{x_d}v\|_{L^2(\Sigma)}+\|q\|_{H^1(\Sigma)}+\|\partial_{x_d}q\|_{L^2(\Sigma)}).$$

We can conclude the proof in the same way as we concluded the proof of inequality (3.10), by considering $\epsilon=e^{-1/h}$ or by using Lemma 3.4 (see remark 3.16): we obtain inequality (3.33) with $\omega\cap\Omega=\psi^{-1}(R(0,z_1))$. □

3.4 Global estimate

In this subsection, we conclude the proofs of Theorems 3.1 and 3.2.

Let us first prove Theorem 3.2. Let $\hat{\omega}$ be a relatively compact open set in Ω. For each $x\in\partial\Omega$, we deduce from Proposition 3.8, that there exist a neighborhood ω_x of x, such that for all $\beta\in\left(0,\frac{1}{2}+\nu\right)$, there exists $c>0$ such that for all $\epsilon>0$ and for all $(u,p)\in H^{\frac{3}{2}+\nu}(\Omega)\times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (1.1), inequality (3.12) is satisfied. We point out that $\partial\Omega\subseteq\bigcup_{x\in\partial\Omega}w_x$ and that $\partial\Omega$ is compact. Thus, we can extract a finite subcover: there exists $N\in\mathbb{N}$ and $x_i\in\partial\Omega$, $i=1,...,N$, such that $\partial\Omega\subset\bigcup_{i=1}^N\omega_{x_i}$. For $i=1,...,N$, let us denote by $\omega_i=\omega_{x_i}$. As a result, we obtain:

$$\forall\,\beta\in\left(0,\frac{1}{2}+\nu\right),\exists\, c>0,\forall\, i\in\{1,...,N\},\forall\,\epsilon>0,\forall\,(u,p)\in H^{\frac{3}{2}+\nu}(\Omega)\times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

$$\|u\|_{H^1(\omega_i\cap\Omega)}+\|p\|_{H^1(\omega_i\cap\Omega)}\le e^{\frac{c}{\epsilon}}\left(\|u\|_{H^1(\hat{\omega})}+\|p\|_{H^1(\hat{\omega})}\right)+\epsilon^\beta\left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)}+\|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}\right).$$

We denote by $\Upsilon=\bigcup_{i=1}^N(\omega_i\cap\Omega)$. Let $r>0$. Let us consider a finite subcover of $\Omega\backslash\overline{\Upsilon}$: there exist $\tilde{N}\in\mathbb{N}$ and $y_i\in\Omega$, $i=1,...,\tilde{N}$ such that $\Omega\backslash\overline{\Upsilon}\subset\bigcup_{i=1}^{\tilde{N}}B(y_i,r)$. For all $i=1,...,\tilde{N}$, up to a decreasing r, $B(y_i,r)$ is a relatively compact open set in Ω where we can apply inequality (3.10) of Proposition 3.6:

$$\exists\, c,s>0,\forall\, i\in\{1,...,\tilde{N}\},\forall\,\epsilon>0,\forall\,(u,p)\in H^1(\Omega)\times H^1(\Omega) \text{ solution of (1.1)},$$

$$\|u\|_{H^1(\mathcal{B}(y_i,r))}+\|p\|_{H^1(\mathcal{B}(y_i,r))}\le\frac{c}{\epsilon}\left(\|u\|_{H^1(\hat{\omega})}+\|p\|_{H^1(\hat{\omega})}\right)+\epsilon^s\left(\|u\|_{H^1(\Omega)}+\|p\|_{H^1(\Omega)}\right).$$

Thus, by summing up the two previous inequalities, taking into account Remark 3.18, we obtain:

$$\forall\, \beta \in \left(0, \frac{1}{2} + \nu\right), \exists\, c > 0, \forall\, \epsilon > 0, \forall\, (u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.1)},$$

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} (\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^{\beta} \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}\right). \tag{3.35}$$

It remains to pass from a relatively compact open set $\hat{\omega}$ to an open set ω (not necessarily relatively compact): we use inequality (3.11) of Proposition 3.6 in order to bound $\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}$ in inequality (3.35) by $\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}$. It directly gives us inequality (3.3) of Theorem 3.2.

Now, if we apply Proposition 3.9, we obtain, ϵ being suitably chosen:

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} &\left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)} \right) \\ &+ \epsilon^{\beta} \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}\right). \end{aligned} \tag{3.36}$$

Let $\theta = \dfrac{1}{1+\nu} \in (0,1)$. Using an interpolation inequality, we obtain that there exists $c > 0$ such that:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left(\|u\|^{1-\theta}_{L^2(\Gamma)} \|u\|^{\theta}_{H^{1+\nu}(\Gamma)} + \|p\|^{1-\theta}_{L^2(\Gamma)} \|p\|^{\theta}_{H^{1+\nu}(\Gamma)} \right).$$

If we write that

$$\|u\|^{1-\theta}_{L^2(\Gamma)} \|u\|^{\theta}_{H^{1+\nu}(\Gamma)} = e^{\frac{2c\theta}{\epsilon}} \|u\|^{1-\theta}_{L^2(\Gamma)} e^{-\frac{2c\theta}{\epsilon}} \|u\|^{\theta}_{H^{1+\nu}(\Gamma)},$$

and

$$\|p\|^{1-\theta}_{L^2(\Gamma)} \|p\|^{\theta}_{H^{1+\nu}(\Gamma)} = e^{\frac{2c\theta}{\epsilon}} \|p\|^{1-\theta}_{L^2(\Gamma)} e^{-\frac{2c\theta}{\epsilon}} \|p\|^{\theta}_{H^{1+\nu}(\Gamma)},$$

according to Young inequality and to the continuity of the trace operator from $H^{\frac{3}{2}+\nu}(\Omega)$ onto $H^{1+\nu}(\Gamma)$, we obtain:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left(e^{\frac{-2c}{\epsilon}} \left(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}\right) + e^{\frac{2c}{\epsilon\nu}} \left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)}\right)\right).$$

Using the fact that $e^{\frac{-c}{\epsilon}} \leq C\epsilon^{\beta}$ for all $\epsilon > 0$, it allows us to replace $\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)}$ in the right hand-side of inequality (3.36) by $\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)}$. This proves inequality (3.2) of Theorem 3.2.

In the same way, using a partition of K by relatively compact open sets and according to inequality (3.11) of Proposition 3.8, we immediately deduce Theorem 3.1.

3.5 Comments

Let us now conclude this section by some comments. By borrowing the approach developed by K. D. Phung in [Phu03], we have thus proved stability estimates stated in Theorem 3.1 and Theorem 3.2 that quantify the unique continuation result of C. Fabre and G. Lebeau in [FL96]. The Carleman estimate that we use near the boundary is a consequence of pseudo-differential calculus. To apply this technique, the domain has to be very regular. In [Bou10], L. Bourgeois proved that the stability estimates proved by K. D. Phung in [Phu03] for $\mathcal{C}^{\infty}$ domains still hold for domains of class $\mathcal{C}^{1,1}$. To do so, he used another technique to derive the same estimates near the boundary: his proof relies on a global Carleman estimate near the boundary on the initial geometry, by following the method of [FI96]. Moreover, in [BD10], L. Bourgeois and J. Dardé complete the results obtained in [Bou10]: they proved a conditional stability estimate related to the ill-posed Cauchy problem for Laplace equation in domain with Lipschitz boundary. For such non smooth domains, difficulties occur when one wants to estimate the function in a neighborhood of $\partial\Omega$: the authors use an interior Carleman estimate and a technique based on a sequence of balls

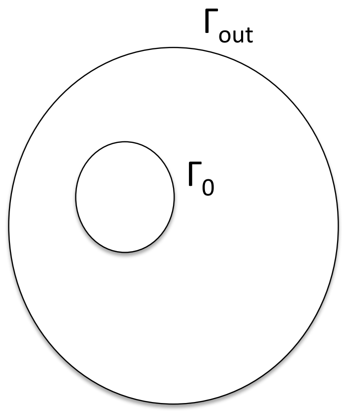

Figure 3: Example of an open set $\Omega \subset \mathbb{R}^2$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_{out} = \emptyset$.

which approaches the boundary, which is inspired by [ABRV00]. Let us emphasize the fact that the inequality obtained by this way is valid for a regular solution u (u belongs to $\mathcal{C}^{1,\alpha}(\Omega)$ and is such that $\Delta u \in L^2(\Omega)$) and that boundary conditions are known on a part of the boundary. These two results suggest that it could be possible to extend the estimates (3.2) and (3.3) to less regular open sets. Another improvement could be to study if the stability estimate of Proposition 3.9 still holds if we have less measurements on the boundary. A possibility could be to use the curl operator instead of the divergence operator as they did in [LUW10] to end up with elliptic equations where the pressure is not involved. Finally, let us remark that we have not used the boundary conditions on the boundary of the domain in the proof of the different stability estimates. A perspective could be to study if our stability estimates can be improved by using the boundary conditions, as in [CR11].

These kinds of stability estimates can be used for different purposes. For example, K. D. Phung uses the stability estimate stated in [Phu03] for the Laplace equation to establish an estimate on the cost of an approximate control function for an elliptic model equation. In [BD10], L. Bourgeois and J. Dardé use stability estimates to study the convergence rate for the method of quasi-reversibility introduced in [LL67] to solve Cauchy problems. As far as we are concerned, we are going to use them to study the inverse problem of identifying a Robin coefficient from measurements available on a part of the boundary in Stokes system: this is the subject of the next section.

4 Application to an inverse problem

We recall that Ω that is a bounded connected subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$. Throughout this section, we assume that $\partial\Omega$ is composed of two sets Γ_0 and Γ_{out} such that $\Gamma_{out} \cup \Gamma_0 = \partial\Omega$ and $\overline{\Gamma}_{out} \cap \overline{\Gamma}_0 = \emptyset$. An example of such geometry in dimension 2 is given in Figure 3.

We consider the following boundary problem:

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \operatorname{div} u & = & 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}. \end{array} \right. \tag{4.1}$$

We want to obtain a stability result for the Robin coefficient q defined on Γ_{out} with respect to the values of u and p on $\Gamma \subseteq \Gamma_0$, where (u, p) is solution of system (4.1).

As explained in the introduction, these models appear in the modeling of biological problems like blood flow in the cardiovascular system or airflow in the lungs. In such systems, Γ_0

corresponds to a physical boundary and Γ_{out} corresponds to an artificial boundary on which Robin boundary conditions or mixed boundary conditions involving the fluid stress tensor and its flux at the outlet are prescribed.

Note that the geometry considered for our problem is a simplification of the ones encountered in blood flow in arteries. Since we need some regularity on the domain as well as regularity of the solution, we restrict ourselves to this kind of assumptions. Nevertheless, as mentioned in Remark 4.8, we could also consider a third part of the boundary where Dirichlet boundary conditions are prescribed and our results still holds in this case.

Let us point out that the uniqueness issue related to our inverse problem has already been studied in [BEG] and is obtained as a consequence of Corollary 1.2. In [BEG], we state that, under some assumptions on the flux g and on the Robin coefficient q, if the velocities are equal on some non-empty open set $\Gamma \subseteq \Gamma_0$, then the Robin coefficients are equal on Γ_{out}.

We introduce the following functional spaces:

$$V = \{v \in H^1(\Omega) / \text{ div } v = 0 \text{ on } \Omega\},$$

and

$$H = \overline{V}^{L^2(\Omega)}.$$

In order to study the Stokes system with Robin boundary conditions, one needs to specify to which space the Robin coefficient q belongs. As stated in Proposition 4.2, we will assume that q belongs to some Sobolev space $H^s(\Gamma_{out})$ where s is large enough so that $qu_{|\Gamma_{out}}$ belongs to $H^r(\Gamma_{out})$ if $u_{|\Gamma_{out}}$ belongs to $H^r(\Gamma_{out})$. This stability in the Sobolev spaces allows to apply regularity result for the Stokes system with Neumann boundary condition. Before stating the regularity result, let us state the following lemma:

Lemma 4.1. *Let $r, s \in \mathbb{R}$, with $s > \frac{d-1}{2}$ and $0 \le r \le s$. Let $q \in H^s(\Gamma_{out})$. The linear operator*

$$\begin{array}{rcl} T : H^r(\Gamma_{out}) & \to & H^r(\Gamma_{out}) \\ u & \mapsto & qu \end{array}$$

is continuous. Furthermore, the following estimate holds true

$$\|qu\|_{H^r(\Gamma_{out})} \le C\|q\|_{H^s(\Gamma_{out})}\|u\|_{H^r(\Gamma_{out})}.$$

Proof of Lemma 4.1. Since $s > \frac{d-1}{2}$, $H^s(\Gamma_{out})$ is a Banach algebra (see [AF03]) and thus $T \in \mathcal{L}(H^s(\Gamma_{out}), H^s(\Gamma_{out}))$ and $\|T\|_s = \sup_{u \in H^s(\Gamma_{out}), u \neq 0} \frac{\|Tu\|_{H^s(\Gamma_{out})}}{\|u\|_{H^s(\Gamma_{out})}} \le \|q\|_{H^s(\Gamma_{out})}$. Moreover, since $H^s(\Gamma_{out}) \hookrightarrow L^\infty(\Gamma_{out})$, $T \in \mathcal{L}(L^2(\Gamma_{out}), L^2(\Gamma_{out}))$ and $\|T\|_0 = \sup_{u \in L^2(\Gamma_{out}), u \neq 0} \frac{\|Tu\|_{L^2(\Gamma_{out})}}{\|u\|_{L^2(\Gamma_{out})}} \le \|q\|_{L^\infty(\Gamma_{out})} \le C\|q\|_{H^s(\Gamma_{out})}$. Thus, the result follows by interpolation (see [BL76] or [Lun09]). $\square$

Let us recall the regularity result proved in [BEG] for the Stokes problem with Robin boundary condition.

Proposition 4.2. *Let $k \in \mathbb{N}$ and $s \in \mathbb{R}$ be such that $s > \frac{d-1}{2}$ and $s \ge \frac{1}{2} + k$. Assume that Ω is of class $C^{k+1,1}$. Let $\alpha > 0$, $M > 0$, $f \in H^k(\Omega)$, $g \in H^{\frac{1}{2}+k}(\Gamma_0)$ and $q \in H^s(\Gamma_{out})$ be such that $\alpha \le q$ on Γ_{out}. Then, the solution (u,p) of system (4.1) belongs to $H^{k+2}(\Omega) \times H^{k+1}(\Omega)$. Moreover, there exists a constant $C(\alpha, M) > 0$ such that for every $q \in H^s(\Gamma_{out})$ satisfying $\|q\|_{H^s(\Gamma_{out})} \le M$,*

$$\|u\|_{H^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \le C(\alpha, M)(\|g\|_{H^{k+\frac{1}{2}}(\Gamma_0)} + \|f\|_{H^k(\Omega)}).$$

In [BCC08], M. Bellassoued, J. Cheng and M. Choulli used a unique continuation estimate for the Laplacian proved by K. D. Phung in [Phu03] to obtain a logarithmic stability estimate for similar inverse problems for the Laplace equation. In the same spirit, we apply the unique continuation estimates for the Stokes system proved below to obtain a logarithmic stability estimate, which is summarized in Theorem 4.3.

Theorem 4.3. *Let $k \in \mathbb{N}^*$ be such that $k+2 > \frac{d}{2}$ and $s \in \mathbb{R}$ be such that $s > \frac{d-1}{2}$ and $s \geq \frac{1}{2}+k$. Let $\Gamma \subseteq \Gamma_0$ be a nonempty open subset of the boundary of Ω. We assume that Γ and Γ_{out} are of class $\mathcal{C}^\infty$. Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$. We assume that $(g, q_j) \in H^{\frac{1}{2}+k}(\Gamma_0) \times H^s(\Gamma_{out})$, for $j = 1, 2$, are such that g is non identically zero, $\|g\|_{H^{\frac{1}{2}+k}(\Gamma_0)} \leq M_1$, $q_j \geq \alpha$ on Γ_{out} and $\|q_j\|_{H^s(\Gamma_{out})} \leq M_2$. We denote by (u_j, p_j) the solution of system (4.1) with $q = q_j$ for $j = 1, 2$. Let K be a compact subset of $\{x \in \Gamma_{out}/u_1 \neq 0\}$ and $m > 0$ be such that $|u_1| \geq m$ on K.*

Then, for all $\beta \in (0,1)$, there exists $C(\alpha, M_1, M_2) > 0$ and $C_1(\alpha, M_1, M_2) > 0$ such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha,M_1,M_2)}{\|u_1-u_2\|_{L^2(\Gamma)}+\|p_1-p_2\|_{L^2(\Gamma)}+\left\|\frac{\partial p_1}{\partial n}-\frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\frac{3}{4}\beta}}. \tag{4.2}$$

Proof of Theorem 4.3. Let us emphasize the fact that, thanks to Proposition 4.2, there exists $C(\alpha, M_1, M_2) > 0$ such that:

$$\|u_j\|_{H^{k+2}(\Omega)} + \|p_j\|_{H^{k+1}(\Omega)} \leq C(\alpha, M_1, M_2), \text{ for } j = 1,\ 2. \tag{4.3}$$

In the following, we denote by $u = u_1 - u_2$ and $p = p_1 - p_2$. We have:

$$(q_2 - q_1)u_1 = q_2 u + \frac{\partial u}{\partial n} - pn, \text{ on } \Gamma_{out}. \tag{4.4}$$

Consequently, since $|u_1| \geq m > 0$ on K:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(M_2) \left(\|u\|_{L^2(K)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(K)} + \|p\|_{L^2(K)} \right). \tag{4.5}$$

Since K and Γ are in $\mathcal{C}^\infty$, we can construct an open set $\omega \subset \Omega$ of class $\mathcal{C}^\infty$ such that $K \subset \partial\omega$ and $\Gamma \subset \partial\omega$. Then, for all $0 < \epsilon < \frac{3}{2}$, using the trace continuity and an interpolation inequality, we have

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leq \frac{1}{m} C(M_2) \left(\|u\|_{H^{3/2+\epsilon}(\omega)} + \|p\|_{L^2(\omega)} \right) \\ &\leq \frac{1}{m} C(M_2) \left(\|u\|^\theta_{H^1(\omega)} \|u\|^{1-\theta}_{H^3(\omega)} + \|p\|_{L^2(\omega)} \right) \end{aligned} \tag{4.6}$$

where $\theta = \dfrac{3}{4}\left(1 - \dfrac{2\epsilon}{3}\right)$. According to inequality (4.3), we then deduce:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(\alpha, M_1, M_2) \left(\|u\|^\theta_{H^1(\omega)} + \|p\|^\theta_{L^2(\omega)} \right).$$

Let $\beta \in (0,1)$ be fixed. We choose $0 < \epsilon < \frac{3}{2}$ small enough such that $\beta' = \frac{\beta}{1-\frac{2\epsilon}{3}}$ belongs to $(0,1)$. We denote by $A = \|u\|_{H^2(\omega)} + \|p\|_{H^2(\omega)}$ and $B = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}$.

Applying inequality (1.3) of Theorem 1.4 with $\nu = \frac{1}{2}$ and with β', we get that there exists $d_0 > 0$ such that for all $\tilde{d} > d_0$, there exists $C(\alpha, M_1, M_2) > 0$, :

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(\alpha, M_1, M_2) \frac{A^\theta}{\left(\ln\left(\tilde{d}\frac{A}{B}\right)\right)^{\beta'\theta}}. \tag{4.7}$$

We conclude by studying the variation of the function defined by $f_y(x) = \dfrac{x}{(\ln(\frac{x}{y}))^{\beta'}}$ on $(y, +\infty)$, for $y = \dfrac{B}{\tilde{d}}$. We have $f_y'(x) = \dfrac{\ln(\frac{x}{y}) - \beta'}{\left(\ln(\frac{x}{y})\right)^{\beta'+1}}$. Let us denote by $x_0 = ye^{\beta'}$. The function f_y is decreasing on $(y, x_0]$ and is increasing on $[x_0, +\infty)$. For $\tilde{d}$ large enough, $A \geq x_0$. Thanks to (4.3) and since f is increasing on $[x_0, +\infty)$, we directly deduce that $f_{\frac{B}{d}}(A) \leq f_{\frac{B}{d}}(C(\alpha, M_1, M_2))$. Using this result in (4.7), we get that there exists $C(\alpha, M_1, M_2) > 0$ and $C_1(\alpha, M_1, M_2) > 0$ such that:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha,M_1,M_2)}{\|u\|_{L^2(\Gamma)}+\|p\|_{L^2(\Gamma)}+\left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)}+\left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\beta'\theta}},$$

and since $\beta'\theta = \frac{3}{4}\beta$ and $\dfrac{\partial u}{\partial n} = pn$ on Γ, this concludes the proof of the theorem. $\square$

Remark 4.4. *Since g is not identically zero, Corollary 1.2 ensures that $\{x \in \Gamma_0 / u_1(x) \neq 0\}$ is not empty. Moreover, according to Proposition 4.2, u_1 is continuous, thus we obtain the existence of a compact K and a constant m as in Theorem 4.3. We notice however that the constants involved in the estimate* (4.2) *and the set K depend on u_1. Finding a uniform lower bound for any solution u of system* (4.1) *remains an open question. We refer to [CJ99], [ADPR03] and [AS06] for the case of the scalar Laplace equation.*

Remark 4.5. *Outside the set K, an estimate of $q_1 - q_2$ may be undetermined or highly unstable. In particular, an estimate of the Robin coefficients on the whole set Γ_{out} might be worst than of logarithmic type (see [BCJ12]). Note however that for a simplified problem, it is in fact possible to obtain a logarithmic stability estimate on the whole set Γ_{out} which does not depend on a given reference solution (see [BEG]).*

Remark 4.6. *In inequality (4.2), the power $\frac{3}{4}\beta$ is directly linked to the regularity of the solution (u, p). If we are more precise in our estimates, we can notice that this power may be improved by a power which depends on k. Indeed, coming back to the inequalities (4.6) and using that $(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$, we get that*

$$\|q_1 - q_2\|_{L^2(K)} \quad \leq \frac{1}{m} C(M_2) \left(\|u\|_{H^1(\omega)}^{\tilde{\theta}} \|u\|_{H^{k+2}(\omega)}^{1-\tilde{\theta}} + \|p\|_{L^2(\omega)} \right)$$

where $\tilde{\theta} = \dfrac{1/2 + k}{1 + k} - \dfrac{\epsilon}{1 + k}$. This estimate allows to obtain the power $\frac{1/2+k}{1+k}\beta$ instead of $\frac{3}{4}\beta$ in inequality (4.2) (when $k = 1$, these powers are equal).

Remark 4.7. *Note that we can still obtain inequality (4.2) by enforcing less regularity on the solution (u, p). In particular, if we consider the case when $d \leq 5$, it is sufficient to assume that (u_j, p_j) belongs to $H^{\frac{5}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ and q_j belongs to $L^\infty(\Gamma_{out})$ and that*

$$\|u_j\|_{H^{\frac{5}{2}+\nu}(\Omega)} + \|p_j\|_{H^{\frac{3}{2}+\nu}(\Omega)} \leq M_1 \text{ and } \|q_j\|_{L^\infty(\Gamma_{out})} \leq M_2,$$

for $j = 1, 2$. In this case, the velocity u_1 is still continuous and with the same reasons as in Remark 4.4, there exist a compact K and a constant $m > 0$ like in Theorem 4.3. Next, instead of the inequalities (4.6)*, we use:*

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leq \frac{1}{m} C(M_2) \left(\|u\|_{H^{3/2+\nu/3}(\omega)} + \|p\|_{L^2(\omega)} \right) \\ &\leq \frac{1}{m} C(M_2) \left(\|u\|_{H^1(\omega)}^{2/3} \|u\|_{H^{5/2+\nu}(\omega)}^{1/3} + \|p\|_{L^2(\omega)} \right) \end{aligned}$$

Then, by performing the same reasoning as above, we get that for all $\beta \in (0, 1)$, there exists $C(\alpha, M_1, M_2) > 0$ and $C_1(\alpha, M_1, M_2) > 0$ such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha,M_1,M_2)}{\|u_1-u_2\|_{L^2(\Gamma)}+\|p_1-p_2\|_{L^2(\Gamma)}+\left\|\frac{\partial p_1}{\partial n}-\frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\frac{2}{3}\beta}}.$$

Let us notice that, due to the fact that the solution is less regular, the power in this inequality is weaker than in inequality (4.2) ($\frac{2}{3}\beta$ instead of $\frac{3}{4}\beta$ for $\beta \in (0, 1)$).

Remark 4.8. *Assume that* $\partial\Omega = \Gamma_0 \cup \Gamma_{out} \cup \Gamma_l$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_{out} = \emptyset$, $\overline{\Gamma}_l \cap \overline{\Gamma}_{out} = \emptyset$ *and* $\overline{\Gamma}_0 \cap \overline{\Gamma}_l = \emptyset$. *Then, Theorem 4.3 remains true for* (u, p) *solution of system*

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}, \end{array}\right.$$

where we have added a homogeneous Dirichlet boundary condition on the part of the boundary Γ_l. *Indeed, for this problem, we still have enough regularity on the solution to perform the same reasoning as in the proof of Theorem 4.3.*

Remark 4.9. *As in [BEG], we can obtain from Theorem 4.3 a stability estimate for the unsteady problem when the Robin coefficient does not depend on time and under assumptions on the asymptotic behavior of the flux* g *when it depends on time. The key idea is to estimate the difference between the solution of the stationary problem and the solution of the non stationary problem by a function which tends to zero as* t *tends to zero, using an inequality coming from semigroup theory. Doing so, the measurements have to be done in infinite time. Let us recall that M. Bellassoued, J. Cheng and M. Choulli have already used this idea in [BCC08] in the case of the Laplace equation with mixed Neumann and Robin boundary conditions.*

The result stated in Theorem 4.3 could be improved in different ways. In the stability estimate (4.2), the Robin coefficients are estimated on a compact subset $K \subset \Gamma_{out}$ which is not a fixed inner portion of Γ_{out} but is unknown and depends on a given reference solution. Up to our knowledge, to obtain an estimate of Robin coefficients on the whole set Γ_{out} or on any compact subset $K \subset \Gamma_{out}$ is still an open question. At last, another natural issue concerns the optimality of the stability estimates: is it possible to obtain better than logarithmic estimates for instance when the Robin coefficient is constant by piece as in [Sin07] for the scalar Laplace equation. This question has been partially answered in [Egl12] by using Theorem 1.3 and the same idea as in [Sin07] leading to holderian stability inequalities.

References

[ABRV00] G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella. Optimal stability for inverse elliptic boundary value problems with unknown boundaries. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(4):755–806, 2000.

[ADPR03] G. Alessandrini, L. Del Piero, and L. Rondi. Stable determination of corrosion by a single electrostatic boundary measurement. *Inverse Problems*, 19(4):973–984, 2003.

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[AS06] G. Alessandrini and E. Sincich. Detecting nonlinear corrosion by electrostatic measurements. *Appl. Anal.*, 85(1-3):107–128, 2006.

[BCC08] M. Bellassoued, J. Cheng, and M. Choulli. Stability estimate for an inverse boundary coefficient problem in thermal imaging. *J. Math. Anal. Appl.*, 343(1):328–336, 2008.

[BCJ12] M. Bellassoued, M. Choulli, and A. Jbalia. Stability of the determination of the surface impedance of an obstacle from the scattering amplitude. *Preprint*, http://hal.archives-ouvertes.fr/hal-00659032/fr/, 2012.

[BD10] L. Bourgeois and J. Dardé. About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains. *Appl. Anal.*, 89(11):1745–1768, 2010.

[BEG] M. Boulakia, A.-C. Egloffe, and C. Grandmont. Stability estimates for a robin coefficient in the two-dimensional stokes system. *Accepted for publication in Mathematical control and related field.*

[BF06] F. Boyer and P. Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

[BGM10] L. Baffico, C. Grandmont, and B. Maury. Multiscale modeling of the respiratory tract. *Math. Models Methods Appl. Sci.*, 20(1):59–93, 2010.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[Bou10] L. Bourgeois. About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains. *M2AN Math. Model. Numer. Anal.*, 44(4):715–735, 2010.

[CCL08] J. Cheng, M. Choulli, and J. Lin. Stable determination of a boundary coefficient in an elliptic equation. *Math. Models Methods Appl. Sci.*, 18(1):107–123, 2008.

[CFJL04] S. Chaabane, I. Fellah, M. Jaoua, and J. Leblond. Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems. *Inverse Problems*, 20(1):47–59, 2004.

[CJ99] S. Chaabane and M. Jaoua. Identification of Robin coefficients by the means of boundary measurements. *Inverse Problems*, 15(6):1425–1438, 1999.

[CR11] P. Cornilleau and L. Robbiano. Carleman estimates for the zaremba boundary condition and stabilization waves. *Submitted*, 2011.

[DL90] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 1.* Springer-Verlag, Berlin, 1990. Physical origins and classical methods, With the collaboration of Philippe Bénilan, Michel Cessenat, André Gervat, Alain Kavenoky and Hélène Lanchon, Translated from the French by Ian N. Sneddon, With a preface by Jean Teillac.

[eEM68] J. L. Lions et E. Magenes. *Problèmes aux limites non homogènes et applications*, volume 1. Dunod, 1968.

[Egl12] A.-C. Egloffe. *Étude de quelques problèmes inverses pour le système de Stokes. Application aux poumons.* PhD thesis, Université Paris VI, 2012.

[FI96] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[FL96] C. Fabre and G. Lebeau. Prolongement unique des solutions de l'equation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.

[GS85] M. Giaquinta and J. Souček. Caccioppoli's inequality and Legendre-Hadamard condition. *Math. Ann.*, 270(1):105–107, 1985.

[Hör85] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[LL67] R. Lattès and J.-L. Lions. *Méthode de quasi-réversibilité et applications.* Travaux et Recherches Mathématiques, No. 15. Dunod, Paris, 1967.

[LR95] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[Lun09] A. Lunardi. *Interpolation theory.* Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, second edition, 2009.

[LUW10] C.-L. Lin, G. Uhlmann, and J.-N. Wang. Optimal three-ball inequalities and quantitative uniqueness for the Stokes system. *Discrete Contin. Dyn. Syst.*, 28(3):1273–1290, 2010.

[Phu03] K.-D. Phung. Remarques sur l'observabilité pour l'équation de laplace. *ESAIM: Control, Optimisation and Calculus of Variations*, 9:621–635, 2003.

[QV03] A. Quarteroni and A. Veneziani. Analysis of a geometrical multiscale model based on the coupling of ODEs and PDEs for blood flow simulations. *Multiscale Model. Simul.*, 1(2):173–195 (electronic), 2003.

[Rob91] L. Robbiano. Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. *Comm. Partial Differential Equations*, 16(4-5):789–800, 1991.

[Sin07] E. Sincich. Lipschitz stability for the inverse Robin problem. *Inverse Problems*, 23(3):1311–1326, 2007.

[VCFJT06] I. E. Vignon-Clementel, C. A. Figueroa, K. E. Jansen, and C. A. Taylor. Outflow boundary conditions for three-dimensional finite element modeling of blood flow and pressure in arteries. *Comput. Methods Appl. Mech. Engrg.*, 195(29-32):3776–3796, 2006.